

Geometric structures associated with the Chern connection attached to a SODE

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Abstract

To each second-order ordinary differential equation σ on a smooth manifold M a G -structure P^σ on $J^1(\mathbb{R}, M)$ is associated and the Chern connection ∇^σ attached to σ is proved to be reducible to P^σ ; in fact, P^σ coincides generically with the holonomy bundle of ∇^σ . The cases of unimodular and orthogonal holonomy are also dealt with. Two characterizations of the Chern connection are given: The first one in terms of the corresponding covariant derivative and the second one as the only principal connection on P^σ with prescribed torsion tensor field. The properties of the curvature tensor field of ∇^σ in relationship to the existence of special coordinate systems for σ are studied. Moreover, all the odd-degree characteristic classes on P^σ are seen to be exact and the usual characteristic classes induced by ∇^σ determine the Chern classes of M . The maximal group of automorphisms of the projection $p: \mathbb{R} \times M \rightarrow \mathbb{R}$ with respect to which ∇^σ has a functorial behaviour, is proved to be the group of p -vertical automorphisms. The notion of a differential invariant under such a group is defined and stated that second-order differential invariants factor through the curvature mapping; a structure is thus established for KCC theory.

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1 Introduction

S.-S. Chern introduced several connections—currently referred to as ‘Chern connections’—in different geometric settings; notably, in: i) the theory of second-order ordinary differential equations [9] in the context of the correspondence between Cartan and Kosambi (cf. [8], [18]), ii) Finsler geometry (see [12] and the references therein), iii) 3-web geometry [10], and iv) almost Hermitian manifolds [11] (the second canonical connection, cf. [23]). These connections usually play a role in modern research on such topics; e.g., see [2], [3], [4], [14], [26], [27], among others.

Due to their importance, Chern classes have eclipsed in part the rest of the works of Chern (cf. [29]); in particular, this happens with the connections above.

In this paper, we consider exclusively the Chern connection of item i).

Let M be a connected C^∞ manifold with natural projections $p: \mathbb{R} \times M \rightarrow \mathbb{R}$ and $p': \mathbb{R} \times M \rightarrow M$. The bundle of r -jets of smooth maps from \mathbb{R} into M is denoted by $p^r: M^r = J^r(\mathbb{R}, M) \rightarrow \mathbb{R}$, with natural projections $p^{rs}: M^r \rightarrow M^s$ for $r > s$, and the r -jet prolongation of the curve $\gamma: \mathbb{R} \rightarrow M$ is denoted by $j^r\gamma: \mathbb{R} \rightarrow M^r$. Every coordinate system (x^i) , $1 \leq i \leq n = \dim M$, on M induces a coordinate system $(t, x^i; x_k^j)$, $i, j = 1, \dots, n$, $0 \leq k \leq r$ on M^r as follows:

$$x_k^j(j_{t_0}^r \gamma) = \frac{d^k(x^j \circ \gamma)}{dt^k}(t_0), \quad x_0^j = x^j.$$

Below, however, for first and second orders we use the more usual ‘dot’ notation; namely, $\dot{x}^j = x_1^j$, and $\ddot{x}^j = x_2^j$.

A second-order ordinary differential equation

$$(1) \quad \ddot{x}^i = F^i(t, x^i, \dot{x}^i), \quad F^i \in C^\infty(M^1), \quad 1 \leq i \leq n,$$

can be better understood as a section $\sigma: M^1 \rightarrow M^2$ of the map $p^{21}: M^2 \rightarrow M^1$ by simply setting $\ddot{x}^i \circ \sigma = F^i$. The correspondence $\sigma \leftrightarrow (F^i)_{i=1}^n$ is natural and bijective.

Remark 1.1. The second-order ordinary differential equation considered in the original paper by Chern [9] (also see [8], [18]) is $\ddot{x}^i + F^i(t, x^i, \dot{x}^i) = 0$, $1 \leq i \leq n$, instead of (1). Hence, in all the formulas below, F^i should be replaced by $-F^i$ in order to compare with the formulas in [9].

Since its introduction in [8, 9, 18], the Chern connection associated to a SODE on M has been studied by several authors; e.g., see [6, 7, 13, 24].

In the sections 3, 4 below, the Chern connection is presented in a similar way as the Levi-Civita connection is introduced in Riemannian Geometry; namely,

1. The notion of a linear frame of M^1 ‘adapted’ to a SODE σ is defined (corresponding to the notion of an orthonormal frame in the Riemannian case).

2. The structure of the set of such frames, is analysed and proved to be a G -structure P^σ of the linear frames of M^1 ; the first prolongation (cf. [16, 21, 22]) of the Lie algebra of the Lie group of P^σ vanishes and the adjoint bundle of P^σ is described in terms of the geometry of σ . The Chern connection ∇^σ is then proved to be reducible to this G -structure.
3. Two characterizations of the Chern connection are provided: The first one characterizes the covariant derivative ∇^σ as a derivation law on the tangent bundle of M^1 and the second one as a principal connection on P^σ with prescribed torsion tensor field. (Observe also the analogy with the Riemannian case.)

In the section 5, given a SODE σ on M , the existence of a fibred coordinate system (t, x^1, \dots, x^n) for the projection $p: M^0 \rightarrow \mathbb{R}$ such that,

$$\ddot{x}^i \circ \sigma = f_0^i(t, x^1, \dots, x^n) + f_j^i(t, x^1, \dots, x^n) \dot{x}^j, \quad 1 \leq i \leq n,$$

is given in terms of the curvature tensor of ∇^σ . The particular cases $f_0^i \equiv 0$ and $f_j^i \equiv 0$ are specially considered.

The odd-degree characteristic classes on P^σ —as defined in [1]—are seen to be exact and the standard characteristic classes induced by ∇^σ are shown to determine the Chern classes of the ground manifold M by means of the natural isomorphism $H^\bullet(M^1; \mathbb{R}) \cong H^\bullet(M; \mathbb{R})$.

In the section 7 the functorial character of the Chern connection is studied. We confine ourselves to consider the group of p -vertical automorphisms of the submersion p , denoted by $\text{Aut}^v(p)$ (see its precise definition at the beginning of this section), as this subgroup is the largest group of transformations for which the functoriality holds. This corresponds to the “problem (B)” in the terminology introduced by Chern in [9]. Chern solves the problem of functoriality of his connection by simply saying (cf. [9, p. 208]): “Il en résulte que la choix $\beta_k^i = \frac{1}{2} \frac{\partial F^i}{\partial y^k}$ a un caractère intrinsèque.”

Finally, the section 8 is devoted to introduce the notion of a differential invariant for SODEs with respect to the group $\text{Aut}^v(p)$. The main result states that invariant functions factor through the curvature mapping induced by the curvature K^σ of the splitting H^σ attached to each SODE σ (see the formulas (11) and (14) below), which almost coincides with the torsion tensor field of the Chern connection ∇^σ , see the formulas (25), (30). This explains the role of the Chern connection in KCC theory. We also remark the similarity between the aforementioned result and the geometric version of the Utiyama theorem (e.g., see [5, 10.2]) in gauge theories.

2 Preliminaries

2.1 Dynamical flows

Every SODE σ defines a vector field $X^\sigma \in \mathfrak{X}(M^1)$, called ‘the dynamical flow’ associated to σ (cf. [24]), as follows: $(X^\sigma)_\xi = (j^1\gamma)_*(d/dt)_{t_0}$, $\forall \xi \in (p^1)^{-1}(t_0)$,

where $\gamma^i = x^i \circ \gamma$, $1 \leq i \leq n$, is the only solution to (1) satisfying the initial conditions $\gamma^i(t_0) = x^i(\xi)$ and $d\gamma^i/dt(t_0) = \dot{x}^i(\xi)$. The local expression of the dynamical flow is $X^\sigma = \partial/\partial t + \dot{x}^i \partial/\partial x^i + F^i \partial/\partial \dot{x}^i$.

2.2 The splitting induced by a SODE

As is known, $p^{10}: M^1 \rightarrow M^0 = \mathbb{R} \times M$ is an affine bundle modelled over p'^*TM ; in fact, given $v \in T_{x_0}M$ and $j_{t_0}^1 \gamma \in M^1$, with $\gamma(t_0) = x_0$, then $v + j_{t_0}^1 \gamma = j_{t_0}^1 \gamma'$ is defined as follows: 1) $\gamma'(t_0) = x_0$, and 2) $\gamma'_*(d/dt)_{t_0} = v + \gamma_*(d/dt)_{t_0}$. Hence, the following exact sequence of vector bundles over M^1 holds:

$$(2) \quad 0 \rightarrow (p' \circ p^{10})^* TM \xrightarrow{\varepsilon} V(p^{10}) \rightarrow TM^1 \xrightarrow{(p^{10})_*} (p^{10})^* TM^0 \rightarrow 0,$$

where $V(p^{10})$ denotes the vector subbundle of p^{10} -vertical vectors and ε is defined by the directional derivative, namely

$$(3) \quad \varepsilon(j_{t_0}^1 \gamma, v)(f) = \lim_{t \rightarrow 0} \frac{f(tv + j_{t_0}^1 \gamma) - f(j_{t_0}^1 \gamma)}{t}, \quad v \in T_{\gamma(t_0)}M, f \in C^\infty(M^1).$$

In local coordinates ε is determined by $\varepsilon(\partial/\partial x^i) = \partial/\partial \dot{x}^i$.

Given a SODE σ , the Lie derivative of the fundamental tensor

$$(4) \quad J = \omega^i \otimes \frac{\partial}{\partial \dot{x}^i}, \quad \omega^i = dx^i - \dot{x}^i dt$$

of M^1 (e.g., see [24, formula (1.13)]) along X^σ is

$$(5) \quad L_{X^\sigma} J = - (dx^i - \dot{x}^i dt) \otimes \frac{\partial}{\partial x^i} + \left\{ \left(\dot{x}^i \frac{\partial F^j}{\partial \dot{x}^i} - F^j \right) dt - \frac{\partial F^j}{\partial \dot{x}^i} dx^i + d\dot{x}^j \right\} \otimes \frac{\partial}{\partial \dot{x}^j},$$

and it is readily checked that $L_{X^\sigma} J$ is diagonalizable with eigenvalues 0, +1, -1, and multiplicities 1, n , n , respectively (cf. [6, p. 6620]). If $T^0(M^1)$, $T^+(M^1)$, $T^-(M^1)$ are the corresponding vector subbundles of eigenvectors, then

$$(6) \quad T(M^1) = T^0(M^1) \oplus T^-(M^1) \oplus T^+(M^1),$$

$$(7) \quad T^0(M^1) = \langle X^\sigma \rangle,$$

$$(8) \quad T^-(M^1) = \langle X_i^\sigma \rangle, \quad X_i^\sigma = \frac{\partial}{\partial x^i} + \frac{1}{2} \frac{\partial F^j}{\partial \dot{x}^i} \frac{\partial}{\partial \dot{x}^j},$$

$$(9) \quad T^+(M^1) = V(p^{10}) = \left\langle \frac{\partial}{\partial \dot{x}^1}, \dots, \frac{\partial}{\partial \dot{x}^n} \right\rangle.$$

Hence, the epimorphism $(p^{10})_*$ in (2) induces an isomorphism

$$(10) \quad (p^{10})_*: T^0(M^1) \oplus T^-(M^1) \xrightarrow{\cong} (p^{10})^* TM^0,$$

whose inverse mapping determines a section $H^\sigma: (p^{10})^*TM^0 \rightarrow TM^1$ of $(p^{10})_*$ given by

$$(11) \quad H^\sigma = dt \otimes X^\sigma + \omega^i \otimes X_i^\sigma,$$

and consequently, the exact sequence (2) splits; i.e., every tangent vector X in TM^1 can uniquely be written as $X = X^v + X^h$, where

$$X^h = H^\sigma((p^{10})_*X) \in T^0(M^1) \oplus T^-(M^1), \quad X^v = X - X^h \in V(p^{10}).$$

In coordinates,

$$(12) \quad \begin{aligned} \left(\frac{\partial}{\partial t}\right)^v &= \left(\frac{1}{2}\dot{x}^i \frac{\partial F^j}{\partial \dot{x}^i} - F^j\right) \frac{\partial}{\partial x^j}, & \left(\frac{\partial}{\partial t}\right)^h &= \frac{\partial}{\partial t} + \left(F^j - \frac{1}{2}\dot{x}^i \frac{\partial F^j}{\partial \dot{x}^i}\right) \frac{\partial}{\partial \dot{x}^j}, \\ \left(\frac{\partial}{\partial x^i}\right)^v &= -\frac{1}{2} \frac{\partial F^j}{\partial \dot{x}^i} \frac{\partial}{\partial \dot{x}^j}, & \left(\frac{\partial}{\partial x^i}\right)^h &= X_i^\sigma, \\ \left(\frac{\partial}{\partial \dot{x}^i}\right)^v &= \frac{\partial}{\partial \dot{x}^i}, & \left(\frac{\partial}{\partial \dot{x}^i}\right)^h &= 0. \end{aligned}$$

Below we need the explicit expression for the curvature form of the splitting H^σ ; i.e.,

$$(13) \quad \begin{aligned} K^\sigma &\in \bigwedge^2 T^*M^1 \otimes V(p^{10}), \\ K^\sigma(X, Y) &= [X^h, Y^h]^v, \quad \forall X, Y \in \mathfrak{X}(M^1). \end{aligned}$$

Proposition 2.1. *The curvature form of the splitting (11) is given as follows:*

$$(14) \quad K^\sigma = -\left(P_j^h dt \wedge \omega^j + \sum_{i < j} T_{ij}^h \omega^i \wedge \omega^j\right) \otimes \frac{\partial}{\partial \dot{x}^h},$$

where the 1-form ω^j is introduced in the formula (4) and

$$(15) \quad T_{ij}^k = \frac{1}{2} \left(\frac{\partial^2 F^k}{\partial x^i \partial \dot{x}^j} - \frac{\partial^2 F^k}{\partial x^j \partial \dot{x}^i} + \frac{1}{2} \left(\frac{\partial F^h}{\partial \dot{x}^i} \frac{\partial^2 F^k}{\partial \dot{x}^h \partial \dot{x}^j} - \frac{\partial F^h}{\partial \dot{x}^j} \frac{\partial^2 F^k}{\partial \dot{x}^h \partial \dot{x}^i} \right) \right),$$

$$(16) \quad P_j^i = \frac{1}{2} X^\sigma \left(\frac{\partial F^i}{\partial \dot{x}^j} \right) - \frac{\partial F^i}{\partial x^j} - \frac{1}{4} \frac{\partial F^k}{\partial \dot{x}^j} \frac{\partial F^i}{\partial \dot{x}^k}.$$

Remark 2.2. The formulas (15), (16) coincide with those in [9, formula (17)] after replacing F^i by $-F^i$, as explained in Remark 1.1.

Proof. As a simple computation shows, the following formulas hold:

$$\begin{aligned} [X^\sigma, X_j^\sigma] &= P_j^i \frac{\partial}{\partial \dot{x}^i} - \frac{1}{2} \frac{\partial F^k}{\partial \dot{x}^j} X_k^\sigma, & \left[X^\sigma, \frac{\partial}{\partial \dot{x}^j} \right] &= -X_j^\sigma - \frac{1}{2} \frac{\partial F^i}{\partial \dot{x}^j} \frac{\partial}{\partial \dot{x}^i}, \\ [X_i^\sigma, X_j^\sigma] &= T_{ij}^k \frac{\partial}{\partial \dot{x}^k}, & \left[X_i^\sigma, \frac{\partial}{\partial \dot{x}^j} \right] &= -\frac{1}{2} \frac{\partial^2 F^h}{\partial \dot{x}^j \partial \dot{x}^i} \frac{\partial}{\partial \dot{x}^h}, \end{aligned}$$

and we can conclude by using Table 12. □

3 G -structure attached to a SODE

Definition 3.1. A linear frame $(X_0, X_1, \dots, X_{2n}) \in F_\xi(M^1)$, $\xi \in M^1$, is said to be *adapted* to a SODE σ on M if it satisfies the following three conditions:

- (i) $X_0 = (X^\sigma)_\xi$.
- (ii) The tangent vector X_{n+i} is p^{10} -vertical for $1 \leq i \leq n$.
- (iii) $X_i = (H^\sigma \circ \varepsilon^{-1})(X_{n+i})$ for $1 \leq i \leq n$.

Proposition 3.2. *Let G be the image of the Lie-group monomorphism*

$$\begin{aligned} \iota: Gl(n, \mathbb{R}) &\rightarrow Gl(2n+1, \mathbb{R}), \\ \iota(\Lambda) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{pmatrix}, \quad \forall \Lambda \in Gl(n, \mathbb{R}). \end{aligned}$$

Let $\pi: F(M^1) \rightarrow M^1$ be the bundle of linear frames of the manifold M^1 . The bundle $\pi: P^\sigma \rightarrow M^1$ of all linear frames adapted to a given SODE σ on M is a G -structure. Moreover, if $(U; x^i)$ is a coordinate open domain in M , then the linear frame

$$(17) \quad \begin{aligned} s: J^1(\mathbb{R}, U) &\rightarrow F(M^1), \\ s(\xi) &= \left((X^\sigma)_\xi, (X_i^\sigma)_\xi, \left(\frac{\partial}{\partial x^i} \right)_\xi \right), \quad 1 \leq i \leq n, \quad \xi \in J^1(\mathbb{R}, U), \end{aligned}$$

defines a section of P^σ , with dual coframe (dt, ω^i, ϖ^i) , $1 \leq i \leq n$, where

$$(18) \quad \varpi^i = dx^i - F^i dt - \frac{1}{2} \frac{\partial F^i}{\partial \dot{x}^j} (dx^j - \dot{x}^j dt).$$

Proof. First of all we prove the last part of the statement, i.e., the section s in the formula (17) takes values in P^σ . In Definition 3.1, the items (i) and (ii) are obvious, and the item (iii) follows directly from the definitions of ε and H^σ in the formulas (3) and (11), respectively. By again taking the items (i)-(iii) in Definition 3.1 into account, it follows that every linear frame $(X_\alpha)_{\alpha=0}^{2n} \in (P^\sigma)_\xi$ can be written as $X_0 = (X^\sigma)_\xi$, $X_j = \lambda_j^i (X_i^\sigma)_\xi$, $X_{n+j} = \lambda_j^i (\partial/\partial x^i)_\xi$, $1 \leq j \leq n$, with $\Lambda = (\lambda_j^i) \in Gl(n, \mathbb{R})$, or equivalently, $(X_0, X_1, \dots, X_{2n}) = s(\xi) \cdot \iota(\Lambda)$, $\iota(\Lambda)$ being the matrix defined in the statement, and the result follows. \square

Remark 3.3. Each G -structure $P \subset F(M^1)$ determines a vector field X_P on M^1 by $(X_P)_\xi = X_0 \in T_\xi M^1$ for every $\xi \in M^1$, where $u = (X_0, X_1, \dots, X_{2n})$ is an arbitrary linear frame in P at ξ . The definition makes sense as X_0 is kept invariant under all the elements in G . A G -structure P is the G -structure associated with a SODE if and only if X_P is a dynamical flow, i.e., $X_P(t) = 1$ and $i_{X_P}(\mathcal{C}_{M^1}) = 0$, where \mathcal{C}_{M^1} is the contact differential system on TM^1 , locally spanned by the 1-forms ω^i , $1 \leq i \leq n$, defined in the formula (4).

Proposition 3.4. *The first prolongation of the Lie algebra $\mathfrak{g} = \iota_* \mathfrak{gl}(n, \mathbb{R})$ of G in Proposition 3.2, vanishes; namely, $\mathfrak{g}^{(1)} = (S^2 V^* \otimes V) \cap (V^* \otimes \mathfrak{g}) = \{0\}$, where $V = \mathbb{R}^{2n+1}$.*

Proof. Let $(v_\alpha)_{\alpha=0}^{2n}$ be the standard basis for V with dual basis $(v^\alpha)_{\alpha=0}^{2n}$. If $t = t_{\alpha\beta}^\gamma v^\alpha \otimes v^\beta \otimes v_\gamma \in \mathfrak{g}^{(1)}$, then, once an index $0 \leq \alpha \leq 2n$ is fixed, the endomorphism $t_{\alpha\beta}^\gamma v^\beta \otimes v_\gamma$ belongs to \mathfrak{g} . Taking the definition of the subalgebra $\mathfrak{g} \subset \mathfrak{gl}(2n+1, \mathbb{R})$ in Proposition 3.2 into account, we obtain $t_{\alpha\beta}^\gamma = 0$, if $\beta = 0$ or $\gamma = 0$, and

$$(19) \quad t_{\alpha\beta}^\gamma = 0, \text{ if } 1 \leq \beta \leq n, n+1 \leq \gamma \leq 2n \text{ or } n+1 \leq \beta \leq 2n, 1 \leq \gamma \leq n,$$

$$(20) \quad t_{\alpha\beta}^\gamma = t_{\alpha, \beta+n}^{\gamma+n}, \quad \beta, \gamma = 1, \dots, n.$$

For $1 \leq \alpha \leq n$ and $\beta, \gamma = 1, \dots, n$, taking $t_{\alpha\beta}^\gamma = t_{\beta\alpha}^\gamma$, (20), and (19) into account, we obtain

$$(21) \quad t_{\alpha\beta}^\gamma \stackrel{(20)}{=} t_{\alpha, \beta+n}^{\gamma+n} = t_{\beta+n, \alpha}^{\gamma+n} \stackrel{(19)}{=} 0.$$

For $n+1 \leq \alpha \leq 2n$ and $\beta, \gamma = 1, \dots, n$, taking $t_{\alpha\beta}^\gamma = t_{\beta\alpha}^\gamma$ and (19) into account, we obtain

$$(22) \quad t_{\alpha\beta}^\gamma = t_{\beta\alpha}^\gamma \stackrel{(19)}{=} 0.$$

Finally, from (20), (21), and (22), we conclude

$$t_{\alpha, \beta+n}^{\gamma+n} = t_{\alpha\beta}^\gamma = 0, \text{ for } 1 \leq \alpha \leq 2n \text{ and } \beta, \gamma = 1, \dots, n.$$

□

Remark 3.5. The previous proof is avoidable by simply looking at the table of Lie algebras with non-trivial first prolongation, e.g., see [22, Tables 1 & 2].

Proposition 3.6. *The adjoint bundle of the G -structure $\pi: P^\sigma \rightarrow M^1$ corresponding to a SODE σ on M can be identified to the vector subbundle*

$$\text{ad}P^\sigma \subseteq T^*(M^1) \otimes T(M^1)$$

of all endomorphisms $E: T(M^1) \rightarrow T(M^1)$ such that,

$$E(T^0(M^1)) = \{0\}, \quad E(T^-(M^1)) \subseteq T^-(M^1), \quad E(T^+(M^1)) \subseteq T^+(M^1),$$

and the composition map $H^\sigma \circ \varepsilon^{-1}$ conjugates $E|_{T^-(M^1)}$ and $E|_{T^+(M^1)}$, i.e.,

$$E|_{T^-(M^1)} \circ H^\sigma \circ \varepsilon^{-1} = H^\sigma \circ \varepsilon^{-1} \circ E|_{T^+(M^1)},$$

where ε , $T^0(M^1)$, $T^-(M^1)$, $T^+(M^1)$, and H^σ are given in the formulas (3), (7), (8), (9), and (11), respectively.

Proof. By its very definition, the adjoint bundle of $\pi: P^\sigma \rightarrow M^1$ is the bundle associated to P^σ with respect to the adjoint representation of G on its Lie algebra \mathfrak{g} , as defined in Propositions 3.2 and 3.4, respectively. The result then follows readily from the conditions (i)-(iii) in Definition 3.1. \square

Proposition 3.7. *The Chern connection as defined in [9] and [24] is reducible (cf. [17, p. 81]) to the G -structure P^σ .*

Proof. As is well known (see [24, Corollary 2.1]), the Chern connection ∇^σ attached to a SODE σ is locally given in the frame $(X_\alpha)_{\alpha=0}^{2n} = (X^\sigma, X_i^\sigma, \partial/\partial \dot{x}^j)$, $i, j = 1, \dots, n$, by the following formulas:

$$(23) \quad \begin{aligned} \nabla_{X^\sigma}^\sigma X^\sigma &= 0, & \nabla_{X^\sigma}^\sigma X_i^\sigma &= -\frac{1}{2} \frac{\partial F^j}{\partial \dot{x}^i} X_j^\sigma, & \nabla_{X^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^i} &= -\frac{1}{2} \frac{\partial F^j}{\partial \dot{x}^i} \frac{\partial}{\partial \dot{x}^j}, \\ \nabla_{X_i^\sigma}^\sigma X^\sigma &= 0, & \nabla_{X_i^\sigma}^\sigma X_j^\sigma &= -\frac{1}{2} \frac{\partial^2 F^k}{\partial \dot{x}^i \partial \dot{x}^j} X_k^\sigma, & \nabla_{X_i^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^j} &= -\frac{1}{2} \frac{\partial^2 F^k}{\partial \dot{x}^i \partial \dot{x}^j} \frac{\partial}{\partial \dot{x}^k}, \\ \nabla_{\frac{\partial}{\partial \dot{x}^i}}^\sigma X^\sigma &= 0, & \nabla_{\frac{\partial}{\partial \dot{x}^i}}^\sigma X_j^\sigma &= 0, & \nabla_{\frac{\partial}{\partial \dot{x}^i}}^\sigma \frac{\partial}{\partial \dot{x}^j} &= 0. \end{aligned}$$

We also set $(\theta^\alpha)_{\alpha=0}^{2n} = (dt, \omega^i, \varpi^i)$, $1 \leq i \leq n$, where ω^i (resp. ϖ^i) is defined in the formula (4) (resp. (18)). Moreover, the equations of the subalgebra $\mathfrak{g} \subset \mathfrak{gl}(2n+1, \mathbb{R})$ (cf. Proposition 3.2) are

$$\begin{aligned} a_0^\alpha &= 0, \quad 0 \leq \alpha \leq 2n, \\ a_\alpha^0 &= 0, \quad 1 \leq \alpha \leq 2n, \\ a_\beta^\alpha &= 0, \quad \text{if } 1 \leq \alpha \leq n, \quad n+1 \leq \beta \leq 2n \text{ or } n+1 \leq \alpha \leq 2n, \quad 1 \leq \beta \leq n, \\ a_\beta^\alpha &= a_{n+\beta}^{n+\alpha}, \quad \alpha, \beta = 1, \dots, n, \end{aligned}$$

where $A = (a_\beta^\alpha)_{\alpha, \beta=0}^{2n} \in \mathfrak{gl}(2n+1, \mathbb{R})$. According to ([15, Proposition 5.4]) Chern's connection is reducible to P^σ if and only if for every $X \in \mathfrak{X}(M^1)$, the following equations hold:

$$(24) \quad \begin{aligned} \theta^0 (\nabla_X^\sigma X_0) &= dt (\nabla_X^\sigma X^\sigma) = 0, \\ \theta^i (\nabla_X^\sigma X_0) &= \omega^i (\nabla_X^\sigma X^\sigma) = 0, \\ \theta^{n+i} (\nabla_X^\sigma X_0) &= \varpi^i (\nabla_X^\sigma X^\sigma) = 0, \\ \theta^0 (\nabla_X^\sigma X_i) &= dt (\nabla_X^\sigma X_i^\sigma) = 0, \\ \theta^0 (\nabla_X^\sigma X_{n+i}) &= dt \left(\nabla_X^\sigma \frac{\partial}{\partial \dot{x}^i} \right) = 0, \\ \theta^{n+i} (\nabla_X^\sigma X_j) &= \varpi^i \left(\nabla_X^\sigma X_j^\sigma \right) = 0, \\ \theta^i (\nabla_X^\sigma X_{n+j}) &= \omega^i \left(\nabla_X^\sigma \frac{\partial}{\partial \dot{x}^j} \right) = 0, \\ \theta^j (\nabla_X^\sigma X_i) &= \theta^{n+j} (\nabla_X^\sigma X_{n+i}). \end{aligned}$$

The equations (24) are an easy consequence of the formulas in Table (23). In fact, the three first conditions are straightforward taking account of the fact that X^σ is parallel. Furthermore, from the equations in Table (23) we obtain $dt(\nabla_X^\sigma X_i^\sigma) = 0$, $\varpi^i(\nabla_X^\sigma X_j^\sigma) = 0$ (resp. $dt(\nabla_X^\sigma \partial/\partial \dot{x}^i) = 0$, $\omega^i(\nabla_X^\sigma \partial/\partial \dot{x}^j) = 0$).

Finally, by setting $X = aX^\sigma + a^i X_i^\sigma + b^i \partial/\partial \dot{x}^i$, we obtain

$$\begin{aligned}\theta^j (\nabla_X^\sigma X_i) &= \omega^j (\nabla_X^\sigma X_i^\sigma) \\ &= -\frac{1}{2} \left(a \frac{\partial F^j}{\partial \dot{x}^i} + a^h \frac{\partial^2 F^j}{\partial \dot{x}^h \partial \dot{x}^i} \right), \\ \theta^{n+j} (\nabla_X^\sigma X_{n+i}) &= \varpi^j \left(\nabla_X^\sigma \frac{\partial}{\partial \dot{x}^i} \right) \\ &= -\frac{1}{2} \left(a \frac{\partial F^j}{\partial \dot{x}^i} + a^h \frac{\partial^2 F^j}{\partial \dot{x}^h \partial \dot{x}^i} \right),\end{aligned}$$

for $i, j = 1, \dots, n$, and the last equation in (24) follows. \square

4 Characterizations of the Chern connection

4.1 First characterization

Theorem 4.1. *Given a SODE σ on M , there is a unique linear connection ∇^σ on M^1 such that,*

- (1) $\nabla^\sigma X^\sigma = 0$.
- (2) $\nabla^\sigma L_{X^\sigma} J = 0$.
- (3) *If $E^\sigma : T(M^1) \rightarrow T(M^1)$ is $E^\sigma = J + H^\sigma \circ \varepsilon^{-1} \circ (L_{X^\sigma} J)^v$, then $\nabla^\sigma E^\sigma = 0$.*
- (4) *The torsion of ∇^σ is the tensor field T^σ given by,*

$$(25) \quad T^\sigma = K^\sigma + dt \wedge (H^\sigma \circ \varepsilon^{-1} \circ (L_{X^\sigma} J)^v),$$

where K^σ is the curvature form of the splitting H^σ induced by σ as defined in the formula (13).

This connection coincides with that defined in [9] and [24].

Proof. By expressing the equation $\nabla^\sigma X^\sigma = 0$ in the frame (17) we obtain

$$(26) \quad (\nabla^\sigma)_{X^\sigma} X^\sigma = 0, \quad (\nabla^\sigma)_{X_i^\sigma} X^\sigma = 0, \quad (\nabla^\sigma)_{\partial/\partial \dot{x}^i} X^\sigma = 0.$$

From item (2), taking the formula (5) for $L_{X^\sigma} J$ and the conditions (26) into

account, we obtain

$$\begin{aligned}
0 &= (\nabla^\sigma L_{X^\sigma} J)(X^\sigma, X^\sigma) = \nabla_{X^\sigma}^\sigma ((L_{X^\sigma} J)(X^\sigma)) - (L_{X^\sigma} J)(\nabla_{X^\sigma}^\sigma X^\sigma), \\
0 &= (\nabla^\sigma L_{X^\sigma} J) \left(\frac{\partial}{\partial \dot{x}^i}, X^\sigma \right) = \nabla_{X^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^i} - (L_{X^\sigma} J) \left(\nabla_{X^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^i} \right), \\
0 &= (\nabla^\sigma L_{X^\sigma} J)(X_i^\sigma, X^\sigma) = -\nabla_{X^\sigma}^\sigma X_i^\sigma - (L_{X^\sigma} J)(\nabla_{X^\sigma}^\sigma X_i^\sigma), \\
0 &= (\nabla^\sigma L_{X^\sigma} J) \left(\frac{\partial}{\partial \dot{x}^i}, X_j^\sigma \right) = \nabla_{X_j^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^i} - (L_{X^\sigma} J) \left(\nabla_{X_j^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^i} \right), \\
0 &= (\nabla^\sigma L_{X^\sigma} J)(X_i^\sigma, X_j^\sigma) = -\nabla_{X_j^\sigma}^\sigma X_i^\sigma - (L_{X^\sigma} J) \left(\nabla_{X_j^\sigma}^\sigma X_i^\sigma \right), \\
0 &= (\nabla^\sigma L_{X^\sigma} J) \left(X_i^\sigma, \frac{\partial}{\partial \dot{x}^j} \right) = -\nabla_{\partial/\partial \dot{x}^j}^\sigma X_i^\sigma - (L_{X^\sigma} J) \left(\nabla_{\partial/\partial \dot{x}^j}^\sigma X_i^\sigma \right), \\
0 &= (\nabla^\sigma L_{X^\sigma} J) \left(\frac{\partial}{\partial \dot{x}^i}, \frac{\partial}{\partial \dot{x}^j} \right) = \nabla_{\partial/\partial \dot{x}^j}^\sigma \frac{\partial}{\partial \dot{x}^i} - (L_{X^\sigma} J) \left(\nabla_{\partial/\partial \dot{x}^j}^\sigma \frac{\partial}{\partial \dot{x}^i} \right).
\end{aligned}$$

Recalling that $T^+(M^1) = \ker(L_{X^\sigma} J - I)$ and $T^-(M^1) = \ker(L_{X^\sigma} J + I)$, from the previous formulas we deduce $\nabla_{X^\sigma}^\sigma \partial/\partial \dot{x}^i$, $\nabla_{X_j^\sigma}^\sigma \partial/\partial \dot{x}^i$, $\nabla_{\partial/\partial \dot{x}^j}^\sigma \partial/\partial \dot{x}^i$ (resp. $\nabla_{X^\sigma}^\sigma X_i^\sigma$, $\nabla_{X_j^\sigma}^\sigma X_i^\sigma$, $\nabla_{\partial/\partial \dot{x}^j}^\sigma X_i^\sigma$) belong to $T^+(M^1)$ (resp. $T^-(M^1)$) and therefore

$$\begin{aligned}
(27) \quad & \nabla_{X^\sigma}^\sigma \partial/\partial \dot{x}^i = A_i^k \frac{\partial}{\partial \dot{x}^k}, \quad \nabla_{X_j^\sigma}^\sigma \partial/\partial \dot{x}^i = A_{ij}^k \partial/\partial \dot{x}^k, \quad \nabla_{\partial/\partial \dot{x}^j}^\sigma \partial/\partial \dot{x}^i = B_{ij}^k \partial/\partial \dot{x}^k, \\
& \nabla_{X^\sigma}^\sigma X_i^\sigma = C_i^k X_k^\sigma, \quad \nabla_{X_j^\sigma}^\sigma X_i^\sigma = C_{ij}^k X_k^\sigma, \quad \nabla_{\partial/\partial \dot{x}^j}^\sigma X_i^\sigma = D_{ij}^k X_k^\sigma.
\end{aligned}$$

Furthermore, according to the definition of E^σ in the statement, its local expression is seen to be

$$(28) \quad E^\sigma = \omega^i \otimes \frac{\partial}{\partial \dot{x}^i} + \varpi^i \otimes X_i^\sigma,$$

where ω^i , ϖ^i , and X_i^σ are given by the formulas (4), (18), and (8), respectively. Hence, from the item (3) and taking the formulas (27) and (28) into account, we obtain

$$\begin{aligned}
0 &= (\nabla^\sigma E^\sigma)(X^\sigma, X^\sigma) = \nabla_{X^\sigma}^\sigma (E^\sigma(X^\sigma)) - E^\sigma(\nabla_{X^\sigma}^\sigma X^\sigma), \\
0 &= (\nabla^\sigma E^\sigma) \left(\frac{\partial}{\partial \dot{x}^i}, X^\sigma \right) = (C_i^k - A_i^k) X_k^\sigma, \\
0 &= (\nabla^\sigma E^\sigma)(X_i^\sigma, X^\sigma) = (A_i^k - C_i^k) \frac{\partial}{\partial \dot{x}^k}, \\
0 &= (\nabla^\sigma E^\sigma) \left(\frac{\partial}{\partial \dot{x}^i}, X_j^\sigma \right) = (C_{ij}^k - A_{ij}^k) X_k^\sigma, \\
0 &= (\nabla^\sigma E^\sigma)(X_i^\sigma, X_j^\sigma) = (A_{ij}^k - C_{ij}^k) \frac{\partial}{\partial \dot{x}^k}, \\
0 &= (\nabla^\sigma E^\sigma) \left(X_i^\sigma, \frac{\partial}{\partial \dot{x}^j} \right) = (B_{ij}^k - D_{ij}^k) \frac{\partial}{\partial \dot{x}^k}, \\
0 &= (\nabla^\sigma E^\sigma) \left(\frac{\partial}{\partial \dot{x}^i}, \frac{\partial}{\partial \dot{x}^j} \right) = (D_{ij}^k - B_{ij}^k) X_k^\sigma.
\end{aligned}$$

Hence $C_i^k = A_i^k$, $C_{ij}^k = A_{ij}^k$, $D_{ij}^k = B_{ij}^k$. Therefore,

$$(29) \quad \begin{aligned} \nabla_{X^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^i} &= A_i^k \frac{\partial}{\partial \dot{x}^k}, & \nabla_{X_j^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^i} &= A_{ij}^k \frac{\partial}{\partial \dot{x}^k}, & \nabla_{\partial/\partial \dot{x}^j}^\sigma \frac{\partial}{\partial \dot{x}^i} &= B_{ij}^k \frac{\partial}{\partial \dot{x}^k}, \\ \nabla_{X^\sigma}^\sigma X_i^\sigma &= A_i^k X_k^\sigma, & \nabla_{X_j^\sigma}^\sigma X_i^\sigma &= A_{ij}^k X_k^\sigma, & \nabla_{\partial/\partial \dot{x}^j}^\sigma X_i^\sigma &= B_{ij}^k X_k^\sigma. \end{aligned}$$

Let us finally impose the condition (4). Taking the expression for K^σ in (14) and the definitions of H^σ , ε^{-1} and $(L_{X^\sigma} J)^v$ in (11), (3) and (5), (12) respectively, the tensor field T^σ defined in (25) is written as:

$$(30) \quad T^\sigma = -P_j^i dt \wedge \omega^j \otimes \frac{\partial}{\partial \dot{x}^i} - \sum_{i < j} T_{ij}^k \omega^i \wedge \omega^j \otimes \frac{\partial}{\partial \dot{x}^k} + dt \wedge \varpi^i \otimes X_i^\sigma.$$

Taking the equations (30) and (26) into account we have

$$\begin{aligned} X_i^\sigma &= T^\sigma \left(X^\sigma, \frac{\partial}{\partial \dot{x}^i} \right) \\ &= \nabla_{X^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^i} - \left(-X_i^\sigma - \frac{1}{2} \frac{\partial F^j}{\partial \dot{x}^i} \frac{\partial}{\partial \dot{x}^j} \right). \end{aligned}$$

Hence

$$(31) \quad \nabla_{X^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^i} = -\frac{1}{2} \frac{\partial F^j}{\partial \dot{x}^i} \frac{\partial}{\partial \dot{x}^j}.$$

Finally, taking (29) into account, from $T^\sigma (X_i^\sigma, \partial/\partial \dot{x}^j) = 0$ we have

$$\begin{aligned} 0 &= T^\sigma \left(X_i^\sigma, \frac{\partial}{\partial \dot{x}^j} \right) \\ &= \nabla_{X_i^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^j} - \nabla_{\partial/\partial \dot{x}^j}^\sigma X_i^\sigma + \frac{1}{2} \frac{\partial^2 F^k}{\partial \dot{x}^j \partial \dot{x}^i} \frac{\partial}{\partial \dot{x}^k} \\ &= \left(A_{ji}^k + \frac{1}{2} \frac{\partial^2 F^k}{\partial \dot{x}^j \partial \dot{x}^i} \right) \frac{\partial}{\partial \dot{x}^k} - B_{ij}^k X_k^\sigma, \end{aligned}$$

and therefore

$$(32) \quad A_{ji}^k = -\frac{1}{2} \frac{\partial^2 F^k}{\partial \dot{x}^j \partial \dot{x}^i}, \quad B_{ij}^k = 0.$$

From (31), (29), and (32) we thus obtain

$$\begin{aligned} \nabla_{X^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^i} &= -\frac{1}{2} \frac{\partial F^j}{\partial \dot{x}^i} \frac{\partial}{\partial \dot{x}^j}, & \nabla_{X_j^\sigma}^\sigma \frac{\partial}{\partial \dot{x}^i} &= -\frac{1}{2} \frac{\partial^2 F^k}{\partial \dot{x}^j \partial \dot{x}^i} \frac{\partial}{\partial \dot{x}^k}, & \nabla_{\partial/\partial \dot{x}^j}^\sigma \frac{\partial}{\partial \dot{x}^i} &= 0, \\ \nabla_{X^\sigma}^\sigma X_i^\sigma &= -\frac{1}{2} \frac{\partial F^j}{\partial \dot{x}^i} X_j^\sigma, & \nabla_{X_j^\sigma}^\sigma X_i^\sigma &= -\frac{1}{2} \frac{\partial^2 F^k}{\partial \dot{x}^j \partial \dot{x}^i} X_k^\sigma, & \nabla_{\partial/\partial \dot{x}^j}^\sigma X_i^\sigma &= 0. \end{aligned}$$

These formulas together with (26) are exactly the same as those in (23). \square

Remark 4.2. From the characterization of the adjoint bundle of the G -structure $\pi: P^\sigma \rightarrow M^1$ given in Proposition 3.6 it follows that the first structure tensor of P^σ , i.e., $T^\sigma \bmod \text{alt}(T^*M^1 \otimes \text{ad}P^\sigma)$ in $\bigwedge^2 T^*M^1 \otimes TM^1 / \text{alt}(T^*M^1 \otimes \text{ad}P^\sigma)$ (e.g. see [16], [21]) never vanishes and that P^σ is not 1-integrable.

Remark 4.3. The item (4) in the theorem can be replaced by the following weaker conditions: $\text{Tor}\nabla^\sigma|_{T^-(M^1) \times T^+(M^1)} = 0$, $i_{X^\sigma} \text{Tor}\nabla^\sigma|_{T^+(M^1)} = H^\sigma \circ \varepsilon^{-1}$.

4.2 Second characterization

Theorem 4.4. *The Chern connection ∇^σ attached to a SODE σ on M is the only linear connection on M^1 reducible to the G -structure $\pi: P^\sigma \rightarrow M^1$ introduced in Proposition 3.2 whose torsion tensor field is given in the formula (25).*

Proof. According to Proposition 3.7 the Chern connection is reducible to P^σ . Furthermore, from the formulas in (23) the torsion of ∇^σ is readily computed, namely,

$$\begin{aligned} \text{Tor}\nabla^\sigma(X^\sigma, X_j^\sigma) &= -P_j^i \frac{\partial}{\partial x^i}, & \text{Tor}\nabla^\sigma(X^\sigma, \frac{\partial}{\partial x^i}) &= X_i^\sigma, \\ \text{Tor}\nabla^\sigma(X_i^\sigma, X_j^\sigma) &= -T_{ij}^k \frac{\partial}{\partial x^k}, & \text{Tor}\nabla^\sigma(X_i^\sigma, \frac{\partial}{\partial x^j}) &= 0, \\ \text{Tor}\nabla^\sigma(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) &= 0, \end{aligned}$$

where the functions P_j^i, T_{ij}^k are defined in the formulas (16), (15), respectively. Hence, from the above equations and the formula (30), one obtains $\text{Tor}\nabla^\sigma = T^\sigma$.

If ∇ is the covariant derivative of another linear connection on M^1 reducible to P^σ , then (e.g., see [21, Proposition I.1]) a unique section h of the vector subbundle $T^*(M^1) \otimes \text{ad}P^\sigma \subseteq T^*(M^1) \otimes T^*(M^1) \otimes T(M^1)$ exists such that $\nabla_X Y = \nabla_X^\sigma Y + h(X, Y)$, $\forall X, Y \in \mathfrak{X}(M^1)$. If the torsion tensor field of ∇ coincides with that of ∇^σ , then h takes values in

$$(S^2 T^*M^1 \otimes TM^1) \cap (T^*M^1 \otimes \text{ad}P^\sigma).$$

But this vector bundle vanishes by virtue of Proposition 3.4. \square

5 Curvature of ∇^σ and characteristic classes

Lemma 5.1. *Let R^σ be the curvature tensor of the Chern connection ∇^σ of a SODE σ on M .*

- (i) *The condition $R^\sigma(X^\sigma, Y)Z = 0$, $\forall Y, Z \in T^-(M^1)$ is equivalent to the condition $R^\sigma(X^\sigma, Y)Z = 0$, $\forall Y \in T^-(M^1), \forall Z \in T^+(M^1)$.*
- (ii) *The condition $R^\sigma(X, Y)Z = 0$, $\forall X, Y, Z \in T^-(M^1)$ is equivalent to the condition $R^\sigma(X, Y)Z = 0$, $\forall X, Y \in T^-(M^1), \forall Z \in T^+(M^1)$.*
- (iii) *The condition $R^\sigma(X, Y)Z = 0$, $\forall X, Z \in T^-(M^1), \forall Y \in T^+(M^1)$ is equivalent to the condition $R^\sigma(X, Y)Z = 0$, $\forall X \in T^-(M^1), \forall Y, Z \in T^+(M^1)$.*

Proof. The result immediately follows from the following formulas:

$$\begin{aligned}
(33) \quad & R^\sigma(X^\sigma, X_j^\sigma) X_i^\sigma = A_{ij}^h X_h^\sigma, \quad R^\sigma(X^\sigma, X_j^\sigma) \frac{\partial}{\partial \dot{x}^i} = A_{ij}^h \frac{\partial}{\partial \dot{x}^h}, \\
& R^\sigma(X_i^\sigma, X_j^\sigma) X_k^\sigma = B_{ijk}^h X_h^\sigma, \quad R^\sigma(X_i^\sigma, X_j^\sigma) \frac{\partial}{\partial \dot{x}^k} = B_{ijk}^h \frac{\partial}{\partial \dot{x}^h} \quad (i < j), \\
& R^\sigma\left(X_i^\sigma, \frac{\partial}{\partial \dot{x}^j}\right) X_k^\sigma = R_{ijk}^h X_h^\sigma, \quad R^\sigma\left(X_i^\sigma, \frac{\partial}{\partial \dot{x}^j}\right) \frac{\partial}{\partial \dot{x}^k} = R_{ijk}^h \frac{\partial}{\partial \dot{x}^h},
\end{aligned}$$

where

$$(34) \quad 2A_{kj}^h = T_{jk}^h - \frac{\partial P_k^h}{\partial \dot{x}^j} - \frac{\partial P_j^h}{\partial \dot{x}^k},$$

$$(35) \quad B_{ijk}^h = -\frac{\partial T_{ij}^h}{\partial \dot{x}^k} \quad (i < j),$$

$$(36) \quad R_{ijk}^h = \frac{1}{2} \frac{\partial^3 F^h}{\partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k},$$

the functions T_{jk}^h , P_k^h being respectively defined in (15), (16), and the rest of components of the curvature tensor vanishes. \square

Remark 5.2. The items (i), (ii), and (iii) above are a simple consequence of the following formula:

$$R^\sigma(X, Y)|_{T^-(M^1)} = H^\sigma \circ \varepsilon^{-1} \circ R^\sigma(X, Y)|_{T^+(M^1)}, \quad \forall X, Y \in T(M^1),$$

which is also deduced from (33).

Theorem 5.3. *Let R^σ be the curvature tensor of the Chern connection ∇^σ of a SODE σ on M .*

- (a) *If one of the equivalent conditions in items (ii) and (iii) of Lemma 5.1 hold on a neighbourhood of a point $x \in M$, then there exists a fibred coordinate system (t, x^i) centred at x on M^0 such that $\ddot{x}^i \circ \sigma$ is polynomial of first degree in $\dot{x}^1, \dots, \dot{x}^n$ for every $1 \leq i \leq n$.*
- (b) *If one of the equivalent conditions in items (i) and (iii) of Lemma 5.1 hold on a neighbourhood of a point $x \in M$, then there exists a fibred coordinate system (t, x^i) centred at x on M^0 such that $\ddot{x}^i \circ \sigma \in C^\infty(M^0)$ for every $1 \leq i \leq n$ (cf. [8, p. 621], [13, Theorem 7]).*
- (c) *If one of the equivalent conditions in item (iii) of Lemma 5.1 holds and K^σ vanishes on a neighbourhood of a point $x \in M$, then there exists a fibred coordinate system (t, x^i) centred at x on M^0 such that $\ddot{x}^i \circ \sigma = 0$ for every $1 \leq i \leq n$ (cf. [8, p. 621], [13, Theorem 6]).*

Proof. According to the formulas (33), the hypothesis in item (a) means $B_{ijk}^h = 0$ and $R_{ijk}^h = 0$. Taking the equations (36) into account, we obtain

$$(37) \quad F^h = F_{ij}^h \dot{x}^i \dot{x}^j + F_i^h \dot{x}^i + F_0^h, \quad F_{ij}^h = F_{ji}^h, \quad F_0^h, F_i^h, F_{ij}^h \in C^\infty(M^0).$$

Substituting (37) into the expression for B_{ijr}^k in (35) and taking the formula (15) into account, we have

$$(38) \quad B_{ijr}^k = \frac{\partial F_{jr}^k}{\partial x^i} - \frac{\partial F_{ir}^k}{\partial x^j} + F_{ir}^h F_{hj}^k - F_{jr}^h F_{hi}^k.$$

For every fixed value $t \in \mathbb{R}$, a symmetric linear connection ∇^t on M can be defined by imposing that its Christoffel symbols in the coordinate system $(x^i)_{i=1}^n$ are $\Gamma_{ij}^h = -F_{ij}^h$, and from the formulas (38) we conclude that all the components of the curvature tensor of ∇^t vanish. Hence there exists a coordinate system $(x'^i)_{i=1}^n$ depending smoothly on t , such that $F_{ij}^h = 0$ and the result follows.

Similarly, the hypothesis in item (b) means $A_{ij}^h = 0$ and $R_{ijk}^h = 0$. Hence the formulas (37) for F^1, \dots, F^n also hold in this case and substituting them into (34) recalling the expressions (15) and (16), we obtain

$$2A_{kj}^h = 2B_{jka}^h \dot{x}^a - 2 \frac{\partial F_{jk}^h}{\partial t} + \frac{\partial F_k^h}{\partial x^j} + F_r^h F_{jk}^r - F_k^r F_{rj}^h.$$

Hence the equations (38) again hold and, in addition, we have

$$(39) \quad 0 = -2 \frac{\partial F_{jk}^h}{\partial t} + \frac{\partial F_j^h}{\partial x^k} + F_r^h F_{jk}^r - F_k^r F_{rj}^h.$$

By virtue of (a) and making a change of coordinates, we can further assume $F_{ij}^h = 0$; hence $F^h = F_i^h \dot{x}^i + F_0^h$, and the equations (39) simply mean that the functions F_j^h are independent of the coordinates x^1, \dots, x^n , i.e., they depend on t only. If we look for a fibred coordinate system $(x'^i)_{i=1}^n$ centred at x on M^0 such that $\dot{x}'^i \circ \sigma \in C^\infty(M^0)$, for $1 \leq i \leq n$, then we obtain

$$x'_{x^a x^b}{}^i = 0, \quad 2x'_{tx^b}{}^i + F_b^j x'_{x^j}{}^i = 0.$$

The first group of equations above is equivalent to saying $x'^i = u_j^i(t) \dot{x}^j + u_0^i(t)$, and from the second group we obtain $\dot{u}_b^i = -\frac{1}{2} F_b^j u_j^i$, which is a system of ordinary differential equations on the unknown functions u_j^i , thus proving (b).

Finally, the tensors K^σ vanishes if, and only if, $P_j^i = 0$ and $T_{ij}^k = 0$, as follows from the expression of K^σ in (14). From $R_{ijk}^h = 0$ we again deduce the equations (37) and substituting them into the formulas (15), (16), and letting $T_{ij}^k = 0$, $P_j^i = 0$, we obtain

$$0 = \left(\frac{\partial F_{jr}^k}{\partial x^i} - \frac{\partial F_{ir}^k}{\partial x^j} + F_{ir}^h F_{hj}^k - F_{jr}^h F_{hi}^k \right) \dot{x}^r + \frac{1}{2} \left(\frac{\partial F_j^k}{\partial x^i} - \frac{\partial F_i^k}{\partial x^j} + F_i^h F_{hj}^k - F_j^h F_{hi}^k \right),$$

$$\begin{aligned}
0 &= \left(\frac{\partial F_{ja}^i}{\partial x^b} - \frac{\partial F_{ab}^i}{\partial x^j} - F_{ja}^k F_{kb}^i + F_{ab}^r F_{jr}^i \right) \dot{x}^a \dot{x}^b \\
&+ \left(\frac{\partial F_{aj}^i}{\partial t} + \frac{1}{2} \frac{\partial F_j^i}{\partial x^a} - \frac{\partial F_a^i}{\partial x^j} - \frac{1}{2} F_k^i F_{aj}^k - \frac{1}{2} F_j^k F_{ak}^i + F_a^h F_{hj}^i \right) \dot{x}^a \\
&+ \frac{1}{2} \frac{\partial F_j^i}{\partial t} - \frac{\partial F_0^i}{\partial x^j} - \frac{1}{4} F_j^k F_k^i + F_0^h F_{hj}^i.
\end{aligned}$$

Hence,

$$(40) \quad 0 = \frac{\partial F_{jr}^k}{\partial x^i} - \frac{\partial F_{ir}^k}{\partial x^j} + F_{ir}^h F_{hj}^k - F_{jr}^h F_{hi}^k,$$

$$(41) \quad 0 = \frac{\partial F_{aj}^i}{\partial t} + \frac{1}{2} \frac{\partial F_j^i}{\partial x^a} - \frac{\partial F_a^i}{\partial x^j} - \frac{1}{2} F_k^i F_{aj}^k - \frac{1}{2} F_j^k F_{ak}^i + F_a^h F_{hj}^i,$$

$$(42) \quad 0 = \frac{1}{2} \frac{\partial F_j^i}{\partial t} - \frac{\partial F_0^i}{\partial x^j} - \frac{1}{4} F_j^k F_k^i + F_0^r F_{jr}^i,$$

$$(43) \quad 0 = \frac{\partial F_{ji}^k}{\partial x^i} - \frac{\partial F_{ij}^k}{\partial x^j} + F_{ij}^h F_{hj}^k - F_{ji}^h F_{hi}^k.$$

Letting $x^0 = t$, an auxiliary symmetric linear connection ∇ can be defined on M^0 by giving its Christoffel symbols as follows:

$$\Gamma_{00}^0 = \Gamma_{0i}^0 = \Gamma_{ij}^0 = 0, \quad \Gamma_{00}^h = -F_0^h, \quad \Gamma_{i0}^h = -\frac{1}{2} F_i^h, \quad \Gamma_{ij}^h = -F_{ij}^h,$$

and, as a computation shows, we obtain

$$R_{0kl}^0 = 0, \quad R_{jkl}^0 = 0, \quad R_{j00}^0 = 0, \quad R_{j0l}^0 = 0,$$

$$R_{0kl}^i = -\frac{1}{2} \frac{\partial F_l^i}{\partial x^k} + \frac{1}{2} \frac{\partial F_k^i}{\partial x^l} + \frac{1}{2} F_l^h F_{hk}^i - \frac{1}{2} F_k^h F_{hl}^i \stackrel{(43)}{=} 0,$$

$$R_{jkl}^i = \frac{\partial F_{kj}^i}{\partial x^l} - \frac{\partial F_{lj}^i}{\partial x^k} + F_{lj}^h F_{kh}^i - F_{kj}^h F_{lh}^i \stackrel{(40)}{=} 0,$$

$$R_{j00}^i = -\frac{1}{2} \frac{\partial F_j^i}{\partial t} + \frac{\partial F_0^i}{\partial x^j} + \frac{1}{4} F_j^h F_h^i - F_0^h F_{hj}^i \stackrel{(42)}{=} 0,$$

$$R_{j0l}^i = -\frac{\partial F_{lj}^i}{\partial t} + \frac{1}{2} \frac{\partial F_j^i}{\partial x^l} + \frac{1}{2} F_{lj}^h F_h^i - \frac{1}{2} F_j^h F_{hl}^i \stackrel{(43)-(41)}{=} 0.$$

Hence ∇ is flat. Consequently, there exists a coordinate system $(x'^0, x'^1, \dots, x'^n)$ on M^0 parallelizing ∇ . Moreover, we have $\nabla(dt) = 0$ (which is equivalent to saying $\Gamma_{00}^0 = \Gamma_{0i}^0 = \Gamma_{ij}^0 = 0$) and hence for all $0 \leq i \leq n$, $X \in \mathfrak{X}(M^0)$,

$$0 = \nabla_X(dt) \left(\frac{\partial}{\partial x'^i} \right) = X \left(dt \left(\frac{\partial}{\partial x'^i} \right) \right) - dt \left(\nabla_X \frac{\partial}{\partial x'^i} \right),$$

thus proving that the function $\partial t / \partial x'^i$ is a constant; accordingly we can further assume $x'^0 = t$, which ends the proof of (c). \square

Remark 5.4. There is a natural and bijective correspondence between homogeneous quadratic SODE σ (i.e., $(\partial/\partial t)^v = 0$ in Table 12) independent of t and symmetric linear connections on M . Actually, given a symmetric linear connection $\tilde{\nabla}$ on M we can define a section $\sigma_{\tilde{\nabla}}: M^1 \rightarrow M^2$ as follows: If $\xi = j_{t_0}^1 \gamma$, then there exists a unique geodesic $\tilde{\gamma}$ for $\tilde{\nabla}$ such that, i) $\tilde{\gamma}(t_0) = \gamma(t_0)$, ii) $\tilde{\gamma}_*(d/dt|_{t_0}) = \gamma_*(d/dt|_{t_0})$. Then $\sigma_{\tilde{\nabla}}(\xi) = j_{t_0}^2 \tilde{\gamma}$. On a coordinate system (x^i) from the equations of geodesics we deduce the equations for $\sigma_{\tilde{\nabla}}$, namely,

$$\ddot{x}^h \circ \sigma_{\tilde{\nabla}} = -\tilde{\Gamma}_{ij}^h \dot{x}^i \dot{x}^j,$$

where $\tilde{\Gamma}_{ij}^h$ are the Christoffel symbols of $\tilde{\nabla}$. Conversely, if $\ddot{x}^h \circ \sigma = F_{ij}^h \dot{x}^i \dot{x}^j$ and (x^i) is another coordinate system overlapping (x^i) , then

$$F_{jk}^h \frac{\partial x'^j}{\partial x^a} \frac{\partial x'^k}{\partial x^b} = \frac{\partial^2 x'^h}{\partial x^a \partial x^b} + \frac{\partial x'^h}{\partial x^i} F_{ab}^i,$$

and this equation is readily seen to be equivalent to the transformation rule of Christoffel's symbols; e.g., see [17, III, Proposition 7.2]. In this case, ∇^σ is completely determined by $\tilde{\nabla}_{\partial/\partial x^i} \partial/\partial x^j = -F_{ij}^h \partial/\partial x^h$, by means of the following formulas:

$$\begin{array}{lll} (\Gamma^\sigma)_{tt}^t = 0, & (\Gamma^\sigma)_{x^i t}^t = 0, & (\Gamma^\sigma)_{\dot{x}^i t}^t = 0, \\ (\Gamma^\sigma)_{tt}^{x^j} = 0, & (\Gamma^\sigma)_{x^h t}^{x^j} = \dot{x}^k F_{hk}^j, & (\Gamma^\sigma)_{\dot{x}^k t}^{x^j} = -\delta_k^j, \\ (\Gamma^\sigma)_{tt}^{\dot{x}^k} = 0, & (\Gamma^\sigma)_{x^i t}^{\dot{x}^k} = F_{hr}^k F_{is}^h \dot{x}^s \dot{x}^r, & (\Gamma^\sigma)_{\dot{x}^i t}^{\dot{x}^k} = -F_{ri}^k \dot{x}^r, \\ (\Gamma^\sigma)_{tx^i}^t = 0, & (\Gamma^\sigma)_{x^j x^i}^t = 0, & (\Gamma^\sigma)_{\dot{x}^j x^i}^t = 0, \\ (\Gamma^\sigma)_{tx^i}^{x^j} = 0, & (\Gamma^\sigma)_{x^k x^i}^{x^j} = -F_{ik}^j, & (\Gamma^\sigma)_{\dot{x}^k x^i}^{x^j} = 0, \\ (\Gamma^\sigma)_{tx^i}^{\dot{x}^j} = 0, & (\Gamma^\sigma)_{x^k x^i}^{\dot{x}^j} = -\left(\frac{\partial F_{ir}^j}{\partial x^k} + F_{ik}^h F_{hr}^j - F_{kh}^j F_{ir}^h\right) \dot{x}^r, & (\Gamma^\sigma)_{\dot{x}^k x^i}^{\dot{x}^j} = -F_{ik}^j, \\ (\Gamma^\sigma)_{t\dot{x}^i}^t = 0, & (\Gamma^\sigma)_{x^j \dot{x}^i}^t = 0, & (\Gamma^\sigma)_{\dot{x}^j \dot{x}^i}^t = 0, \\ (\Gamma^\sigma)_{t\dot{x}^i}^{x^k} = 0, & (\Gamma^\sigma)_{x^j \dot{x}^i}^{x^k} = 0, & (\Gamma^\sigma)_{\dot{x}^j \dot{x}^i}^{x^k} = 0, \\ (\Gamma^\sigma)_{t\dot{x}^i}^{\dot{x}^j} = 0, & (\Gamma^\sigma)_{x^k \dot{x}^i}^{\dot{x}^j} = -F_{ik}^j, & (\Gamma^\sigma)_{\dot{x}^k \dot{x}^i}^{\dot{x}^j} = 0, \end{array}$$

as follows from the formulas in (23) for this particular case. By using these formulas and the identification $M^1 \cong \mathbb{R} \times TM$, $j_{t_0}^1 \gamma \mapsto (t_0, \gamma_*(d/dt)_{t_0})$, from [28, II, formulas (7.8)], one realizes that the component of ∇^σ in the tangent bundle coincides with the horizontal lift $\tilde{\nabla}^H$ of $\tilde{\nabla}$.

Finally, the components of the torsion and curvature tensor fields of ∇^σ are expressed in terms of the components of the curvature tensor field of the connection $\tilde{\nabla}$ by means of the following formulas: $T_{ij}^k = \tilde{R}_{rji}^k \dot{x}^r$, $P_j^h = \tilde{R}_{sjr}^h \dot{x}^r \dot{x}^s$, $A_{kj}^h = \tilde{R}_{krj}^h \dot{x}^r$, $B_{ijk}^h = \tilde{R}_{kij}^h$.

Let V be \mathbb{R}^{2n+1} with basis $(v_i)_{i=1}^{2n+1}$ and dual basis $(v^i)_{i=1}^{2n+1}$. In [1, 3.4] a generalized Chern-Weil homomorphism $(S(\mathfrak{g}^*) \otimes \wedge V^*)^G \rightarrow \Omega(M)$ has been defined for every G -structure on M . In our case, $G \cong Gl(n, \mathbb{R})$, $\mathfrak{g} \cong \mathfrak{gl}(n, \mathbb{R})$, and, as calculation shows, $(S(\mathfrak{g}^*) \otimes \wedge (V^*))^G = S(\mathfrak{g}^*)^G \oplus (S(\mathfrak{g}^*)^G \otimes v^1)$. In

fact, every $t \in S^a(\mathfrak{g}^*) \otimes \bigwedge^b(V^*)$ can be written as,

$$\begin{aligned} t = & \sum_{2 \leq i_2 < \dots < i_b \leq 2n+1} s_{a,I} \otimes v^1 \wedge v^{i_2} \wedge \dots \wedge v^{i_b} \\ & + \sum_{2 \leq j_1 < \dots < j_b \leq 2n+1} s_{a,J} \otimes v^{j_1} \wedge v^{j_2} \wedge \dots \wedge v^{j_b}, \end{aligned}$$

where $I = (i_2, \dots, i_b) \in \mathbb{N}^{b-1}$, $J = (j_1, \dots, j_b) \in \mathbb{N}^b$, $s_{a,I}, s_{a,J} \in S^a(\mathfrak{g}^*)$.

If $A \in G = \iota(Gl(n, \mathbb{R})) \subset Gl(2n+1, \mathbb{R})$ is the matrix in Proposition 3.2 corresponding to $\Lambda = \lambda I_n$, $\lambda \in \mathbb{R}^*$, i.e., $A \cdot v^1 = v^1$, $A \cdot v^i = \lambda v^i$, $2 \leq i \leq 2n+1$, then the invariance equation $A \cdot t - t = 0$ is equivalent to saying,

$$\begin{aligned} 0 = & \sum_{2 \leq i_2 < \dots < i_b \leq 2n+1} (\lambda - \lambda^b) s_{a,I} \otimes v^1 \wedge v^{i_2} \wedge \dots \wedge v^{i_b} \\ & + \sum_{2 \leq j_1 < \dots < j_b \leq 2n+1} (1 - \lambda^b) s_{a,J} \otimes v^{j_1} \wedge v^{j_2} \wedge \dots \wedge v^{j_b}, \end{aligned}$$

and the result readily follows.

Accordingly, if θ is the soldering form on $F(M^1)$, Ω^σ is the curvature form of the Chern connection attached to σ , and $\theta' = \theta|_{P^\sigma}$, $\Omega'^\sigma = \Omega^\sigma|_{P^\sigma}$ (cf. [17, II, Proposition 6.1-(b)]) are their restrictions to P^σ respectively, then for every Weil polynomial $f \in S(\mathfrak{g}^*)^G$ we obtain $f(\Omega'^\sigma) \wedge \theta'^1 = d(tf(\Omega'^\sigma))$.

In summary, the Chern connection attached to an arbitrary SODE σ on M determines the Chern classes on M in the standard Chern-Weil homomorphism (under the natural isomorphism $H^\bullet(M^1; \mathbb{R}) \cong H^\bullet(M; \mathbb{R})$), whereas the characteristic forms of odd degree attached to P^σ are all exact. This also shows that the sufficient condition on the connection ω of being symmetric in [1, 3.4, Theorem] is not necessary.

6 Holonomy of ∇^σ

6.1 General holonomy

Proposition 6.1. *Assume M and σ are of class C^ω . If the equation*

$$\begin{aligned} 0 = 2R^\sigma(X, Y)(U) + R^\sigma((H^\sigma \circ \varepsilon^{-1})Z, T)(U) \\ + R^\sigma((H^\sigma \circ \varepsilon^{-1})T, Z)(U), \end{aligned}$$

$X, Y \in T_\xi^- M^1$, $Z, T, U \in T_\xi^+ M^1$, implies $X = Y = Z = T = U = 0$, then the holonomy algebra of ∇^σ is $\mathfrak{gl}(n, \mathbb{R})$.

Proof. According to the hypothesis in the statement, the holonomy algebra can be computed from the infinitesimal holonomy algebra (see [17, II, Theorem 10.8]). Moreover, from [17, III, Theorem 9.2] we know that such algebra is spanned by the endomorphisms

$$\left((\nabla^\sigma)^k R^\sigma \right) (X, Y; V_1; \dots; V_k), \quad \forall X, Y, V_1, \dots, V_k \in T_\xi M^1, \quad \forall k \in \mathbb{N}.$$

For $k = 0$, from the formulas (33) we obtain

$$(44) \quad R^\sigma(X_i^\sigma, X_j^\sigma) = B_{ijk}^h \left(\omega^k \otimes X_h^\sigma + \varpi^k \otimes \frac{\partial}{\partial \dot{x}^h} \right), \quad i < j,$$

$$(45) \quad R^\sigma \left(X_i^\sigma, \frac{\partial}{\partial \dot{x}^j} \right) = R_{ijk}^h \left(\omega^k \otimes X_h^\sigma + \varpi^k \otimes \frac{\partial}{\partial \dot{x}^h} \right), \quad i \leq j.$$

Hence

$$\begin{aligned} R^\sigma(X_i^\sigma, X_j^\sigma) \Big|_{T^+(M^1)} &= B_{ijk}^h \varpi^k \otimes \frac{\partial}{\partial \dot{x}^h}, \quad i < j, \\ R^\sigma \left(X_i^\sigma, \frac{\partial}{\partial \dot{x}^j} \right) \Big|_{T^+(M^1)} &= R_{ijk}^h \varpi^k \otimes \frac{\partial}{\partial \dot{x}^h} \quad i \leq j. \end{aligned}$$

If the matrices $\left(B_{ijk}^h(\xi) \right)_{h,k=1}^n$, $i < j$, $\left(R_{ijk}^h(\xi) \right)_{h,k=1}^n$, $i \leq j$ span $\mathfrak{gl}(n, \mathbb{R})$, we can conclude. Moreover, if

$$\Upsilon^\sigma : \bigwedge^2 T^-(M^1) \oplus S^2 T^+(M^1) \rightarrow \text{End} T^+(M^1)$$

is the homomorphism given by,

$$\begin{aligned} \Upsilon^\sigma(X \wedge Y, Z \odot T)(U) &= R^\sigma(X, Y)(U) \\ &+ \frac{1}{2} \{ R^\sigma((H^\sigma \circ \varepsilon^{-1})Z, T)(U) + R^\sigma((H^\sigma \circ \varepsilon^{-1})T, Z)(U) \}, \end{aligned}$$

then the condition in the statement is readily seen to be equivalent to saying Υ^σ is an isomorphism, as its matrix on the bases

$$\left(X_i^\sigma \wedge X_j^\sigma, \frac{\partial}{\partial \dot{x}^h} \odot \frac{\partial}{\partial \dot{x}^k} \right), \quad i < j, h \leq k; \quad \left(\varpi^a \otimes \frac{\partial}{\partial \dot{x}^b} \right), \quad a, b = 1, \dots, n,$$

of $\bigwedge^2 T^-(M^1) \oplus S^2 T^+(M^1)$, $\text{End} T^+(M^1) = T^+(M^1)^* \otimes T^+(M^1)$, respectively, is $\left(\left(B_{ijb}^a \right)_{a,b=1}^n, \left(R_{hkb}^a \right)_{a,b=1}^n \right)$. \square

6.2 Special holonomy

Next, we determine the conditions under which the holonomy algebra of ∇^σ is contained in $\mathfrak{sl}(n, \mathbb{R})$. If M and σ still are of class C^ω , then according to ([17, Lemma 1, p. 152]), the holonomy algebra is spanned by the endomorphisms

$$(46) \quad \nabla_{V_l}^\sigma \cdots \nabla_{V_1}^\sigma (R^\sigma(X, Y)), \quad \forall X, Y, V_1, \dots, V_l \in \mathfrak{X}(M^1), \quad \forall l \in \mathbb{N}.$$

Lemma 6.2. *For every system of vector fields $X, Y, V_1, \dots, V_l \in \mathfrak{X}(M^1)$, $l \in \mathbb{N}$, the endomorphism $\nabla_{V_l}^\sigma \cdots \nabla_{V_1}^\sigma (R^\sigma(X, Y))$ can locally be written as follows:*

$$S_k^h \left(\omega^k \otimes X_h^\sigma + \varpi^k \otimes \frac{\partial}{\partial \dot{x}^h} \right), \quad S_k^h \in C^\infty(M^1).$$

Proof. For $l = 0$ from the formulas (33) we obtain

$$R^\sigma(X^\sigma, X_i^\sigma) = A_{ki}^h \left(\omega^k \otimes X_h^\sigma + \varpi^k \otimes \frac{\partial}{\partial \dot{x}^h} \right),$$

which, together with the formulas (44) and (45), prove the statement in this case. For $l \geq 1$ the proof is by induction. If

$$(\nabla^\sigma)_{V_{l-1}}^\sigma \cdots \nabla_{V_1}^\sigma (R^\sigma(X, Y)) = S_k^h \left(\omega^k \otimes X_h^\sigma + \varpi^k \otimes \frac{\partial}{\partial \dot{x}^h} \right),$$

then,

$$\begin{aligned} \nabla_{X^\sigma}^\sigma \left(\nabla_{V_{l-1}}^\sigma \cdots \nabla_{V_1}^\sigma (R^\sigma(X, Y)) \right) &= S_{0,k}^h \left(\omega^k \otimes X_h^\sigma + \varpi^k \otimes \frac{\partial}{\partial \dot{x}^h} \right), \\ \nabla_{X_j^\sigma}^\sigma \left(\nabla_{V_{l-1}}^\sigma \cdots \nabla_{V_1}^\sigma (R^\sigma(X, Y)) \right) &= S_{j,k}^h \left(\omega^k \otimes X_h^\sigma + \varpi^k \otimes \frac{\partial}{\partial \dot{x}^h} \right), \\ \nabla_{\frac{\partial}{\partial \dot{x}^j}}^\sigma \left(\nabla_{V_{l-1}}^\sigma \cdots \nabla_{V_1}^\sigma (R^\sigma(X, Y)) \right) &= \frac{\partial S_k^h}{\partial \dot{x}^j} \left(\omega^k \otimes X_h^\sigma + \varpi^k \otimes \frac{\partial}{\partial \dot{x}^h} \right), \end{aligned}$$

where

$$\begin{aligned} S_{0,k}^h &= X^\sigma(S_k^h) + \frac{1}{2} S_r^h \frac{\partial F^r}{\partial \dot{x}^k} - \frac{1}{2} S_k^r \frac{\partial F^h}{\partial \dot{x}^r}, \\ S_{j,k}^h &= X_j^\sigma(S_k^h) + \frac{1}{2} S_r^h \frac{\partial^2 F^r}{\partial \dot{x}^j \partial \dot{x}^k} - \frac{1}{2} S_k^r \frac{\partial^2 F^h}{\partial \dot{x}^j \partial \dot{x}^r}. \end{aligned}$$

□

By induction on l the endomorphisms (46) are proved to be traceless if and only if there exists a function $F \in C^\infty(M^0)$ such that,

$$(47) \quad \sum_{h=1}^n \frac{\partial F^h}{\partial \dot{x}^h} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x^i} \dot{x}^i.$$

For $l = 0$ taking the formula (36) into account, we obtain

$$\text{tr} (R_{ijk}^h)_{h,k=1}^n = \frac{\partial^2}{\partial \dot{x}^i \partial \dot{x}^j} \left(\sum_{h=1}^n \frac{\partial F^h}{\partial \dot{x}^h} \right), \quad 1 \leq i \leq j \leq n.$$

Hence the matrices $(R_{ijk}^h)_{h,k=1}^n$ are traceless, if and only if,

$$\sum_{h=1}^n \partial F^h / \partial \dot{x}^h = F_0 + F_i \dot{x}^i, \quad F_0, F_i \in C^\infty(M^0).$$

Furthermore, taking the formula above and the identity

$$\begin{aligned} 2B_{ijk}^h &= -\frac{\partial^3 F^h}{\partial x^i \partial \dot{x}^j \partial \dot{x}^k} - \frac{1}{2} \frac{\partial F^l}{\partial \dot{x}^i} \frac{\partial^3 F^h}{\partial \dot{x}^j \partial \dot{x}^k \partial \dot{x}^l} + \frac{\partial^3 F^h}{\partial x^j \partial \dot{x}^i \partial \dot{x}^k} \\ &+ \frac{1}{2} \frac{\partial F^l}{\partial \dot{x}^j} \frac{\partial^3 F^h}{\partial \dot{x}^i \partial \dot{x}^k \partial \dot{x}^l} + \frac{1}{2} \frac{\partial^2 F^r}{\partial \dot{x}^j \partial \dot{x}^k} \frac{\partial^2 F^h}{\partial \dot{x}^i \partial \dot{x}^r} - \frac{1}{2} \frac{\partial^2 F^r}{\partial \dot{x}^i \partial \dot{x}^k} \frac{\partial^2 F^h}{\partial \dot{x}^j \partial \dot{x}^r} \end{aligned}$$

into account, we conclude that the matrices $(B_{ijk}^h)_{h,k=1}^n$ are traceless if and only if, $\partial F_j / \partial x^i = \partial F_i / \partial x^j$ for $1 \leq i < j \leq n$. Finally, from the identity

$$2A_{kj}^h = -\frac{\partial^3 F^h}{\partial t \partial \dot{x}^k \partial \dot{x}^j} - \dot{x}^r \frac{\partial^3 F^h}{\partial x^r \partial \dot{x}^k \partial \dot{x}^j} - F^r \frac{\partial^3 F^h}{\partial \dot{x}^r \partial \dot{x}^k \partial \dot{x}^j} + \frac{\partial^2 F^h}{\partial x^j \partial \dot{x}^k} \\ + \frac{1}{2} \frac{\partial F^h}{\partial \dot{x}^r} \frac{\partial^2 F^r}{\partial \dot{x}^k \partial \dot{x}^j} - \frac{1}{2} \frac{\partial F^r}{\partial \dot{x}^k} \frac{\partial^2 F^h}{\partial \dot{x}^r \partial \dot{x}^j},$$

we deduce that the matrices $(A_{kj}^h)_{h,k=1}^n$ are traceless for $1 \leq j \leq n$ if and only if, $\partial F_j / \partial t = \partial F_0 / \partial x^j$.

For $l \geq 1$, by applying Lemma 6.2 and the induction hypothesis, we obtain

$$\nabla_{V_{l-1}}^\sigma \cdots \nabla_{V_1}^\sigma (R^\sigma(X, Y)) = S_k^h \left(\omega^k \otimes X_h^\sigma + \varpi^k \otimes \frac{\partial}{\partial \dot{x}^h} \right), \quad S_h^h = 0.$$

Hence, taking the formulas for $S_{0,k}^h$ and $S_{j,k}^h$ at the end of the proof of Lemma 6.2 into account we obtain

$$\text{tr} (S_{0,k}^h)_{h,k=1}^n = X^\sigma \left(\text{tr} (S_k^h)_{h,k=1}^n \right) + \frac{1}{2} \left(S_r^h \frac{\partial F^r}{\partial \dot{x}^h} - S_h^r \frac{\partial F^h}{\partial \dot{x}^r} \right) = 0, \\ \text{tr} (S_{j,k}^h)_{h,k=1}^n = X_j^\sigma \left(\text{tr} (S_k^h)_{h,k=1}^n \right) + \frac{1}{2} \left(S_r^h \frac{\partial^2 F^r}{\partial \dot{x}^j \partial \dot{x}^h} - S_h^r \frac{\partial^2 F^h}{\partial \dot{x}^j \partial \dot{x}^r} \right) = 0, \\ \text{tr} \left(\frac{\partial S_k^h}{\partial \dot{x}^j} \right)_{h,k=1}^n = \frac{\partial}{\partial \dot{x}^j} \left(\text{tr} (S_k^h)_{h,k=1}^n \right) = 0.$$

As a calculation shows, the equation (47) means that the volume form

$$\exp(-F) dt \wedge \omega^1 \wedge \dots \wedge \omega^n \wedge \varpi^1 \wedge \dots \wedge \varpi^n$$

is parallel with respect to ∇^σ .

6.3 Orthogonal holonomy

Proposition 6.3. *The holonomy group of ∇^σ is contained in $SO(n)$ if and only if for every $\xi \in M^1$ there exist a coordinate neighbourhood $(N^0; t, x^i)$ of $p^{10}(\xi)$ in M^0 and a positive definite symmetric matrix $U = (u_j^i)_{i,j=1}^n$, $u_j^i \in C^\infty(N^0)$, such that the following equation holds:*

$$(48) \quad \frac{\partial U}{\partial t} + \dot{x}^i \frac{\partial U}{\partial x^i} + UW + (UW)^t = 0,$$

where $W = (w_j^i)$, $w_j^i = \frac{1}{2} \frac{\partial F^i}{\partial \dot{x}^j}$.

Proof. The holonomy group of ∇^σ is contained in $SO(n)$ if and only if there exists a Riemannian metric g^1 on M^1 such that,

- (i) g^1 is parallel with respect to ∇^σ .

- (ii) For every $\xi \in M^1$ there exist an open neighbourhood N^1 and a section of P^σ defined over N^1 , $(X^\sigma, \bar{X}_j^\sigma = \Lambda_j^i X_i^\sigma, \bar{X}_j = \Lambda_j^i \partial/\partial \dot{x}^i)_{j=1}^n, \Lambda_j^i \in C^\infty(N^1)$, which is an orthonormal linear frame with respect to g^1 ; e.g., see [17, II, §7, Lemma 2; III, Proposition 1.5; IV, Proposition 2.1].

We impose the condition $\nabla^\sigma g^1 = 0$, by using the basis $(X^\sigma, X_i^\sigma, \partial/\partial \dot{x}^i)_{i=1}^n$. From the identities

$$(49) \quad \begin{aligned} g^1(X^\sigma, X_k^\sigma) &= g^1(X^\sigma, (\Lambda^{-1})_k^a \Lambda_a^r X_r^\sigma) = (\Lambda^{-1})_k^a g^1(X^\sigma, \bar{X}_a^\sigma) = 0, \\ g^1(X^\sigma, \partial/\partial \dot{x}^k) &= (\Lambda^{-1})_k^a g^1(X^\sigma, \Lambda_a^r \partial/\partial \dot{x}^r) = (\Lambda^{-1})_k^a g^1(X^\sigma, \bar{X}_a) = 0, \\ g^1(X_h^\sigma, X_k^\sigma) &= (\Lambda^{-1})_h^b g^1(\bar{X}_b^\sigma, \bar{X}_a^\sigma) (\Lambda^{-1})_k^a = (\Lambda^{-1})_h^b \delta_{ba} (\Lambda^{-1})_k^a, \\ g^1(X_h^\sigma, \partial/\partial \dot{x}^k) &= (\Lambda^{-1})_h^b g^1(\bar{X}_b^\sigma, \bar{X}_a) (\Lambda^{-1})_k^a = 0, \\ g^1(\partial/\partial \dot{x}^h, \partial/\partial \dot{x}^k) &= (\Lambda^{-1})_h^b g^1(\bar{X}_b, \bar{X}_a) (\Lambda^{-1})_k^a = (\Lambda^{-1})_h^b \delta_{ba} (\Lambda^{-1})_k^a, \end{aligned}$$

and the formulas in (23), we conclude the following equations hold identically:

$$\begin{aligned} X^\sigma(g^1(X^\sigma, X^\sigma)) &= g^1(\nabla_{X^\sigma}^\sigma X^\sigma, X^\sigma) + g^1(X^\sigma, \nabla_{X^\sigma}^\sigma X^\sigma), \\ X^\sigma(g^1(X^\sigma, X_i^\sigma)) &= g^1(\nabla_{X^\sigma}^\sigma X^\sigma, X_i^\sigma) + g^1(X^\sigma, \nabla_{X^\sigma}^\sigma X_i^\sigma), \\ X^\sigma(g^1(X^\sigma, \partial/\partial \dot{x}^j)) &= g^1(\nabla_{X^\sigma}^\sigma X^\sigma, \partial/\partial \dot{x}^j) + g^1(X^\sigma, \nabla_{X^\sigma}^\sigma \partial/\partial \dot{x}^j), \\ X^\sigma(g^1(X_i^\sigma, \partial/\partial \dot{x}^j)) &= g^1(\nabla_{X^\sigma}^\sigma X_i^\sigma, \partial/\partial \dot{x}^j) + g^1(X_i^\sigma, \nabla_{X^\sigma}^\sigma \partial/\partial \dot{x}^j), \end{aligned}$$

$$\begin{aligned} X_k^\sigma(g^1(X^\sigma, X^\sigma)) &= g^1(\nabla_{X_k^\sigma}^\sigma X^\sigma, X^\sigma) + g^1(X^\sigma, \nabla_{X_k^\sigma}^\sigma X^\sigma), \\ X_k^\sigma(g^1(X^\sigma, X_i^\sigma)) &= g^1(\nabla_{X_k^\sigma}^\sigma X^\sigma, X_i^\sigma) + g^1(X^\sigma, \nabla_{X_k^\sigma}^\sigma X_i^\sigma), \\ X_k^\sigma(g^1(X^\sigma, \partial/\partial \dot{x}^j)) &= g^1(\nabla_{X_k^\sigma}^\sigma X^\sigma, \partial/\partial \dot{x}^j) + g^1(X^\sigma, \nabla_{X_k^\sigma}^\sigma \partial/\partial \dot{x}^j), \\ X_k^\sigma(g^1(X_i^\sigma, \partial/\partial \dot{x}^j)) &= g^1(\nabla_{X_k^\sigma}^\sigma X_i^\sigma, \partial/\partial \dot{x}^j) + g^1(X_i^\sigma, \nabla_{X_k^\sigma}^\sigma \partial/\partial \dot{x}^j), \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \dot{x}^k}(g^1(X^\sigma, X^\sigma)) &= g^1(\nabla_{\partial/\partial \dot{x}^k}^\sigma X^\sigma, X^\sigma) + g^1(X^\sigma, \nabla_{\partial/\partial \dot{x}^k}^\sigma X^\sigma), \\ \frac{\partial}{\partial \dot{x}^k}(g^1(X^\sigma, X_i^\sigma)) &= g^1(\nabla_{\partial/\partial \dot{x}^k}^\sigma X^\sigma, X_i^\sigma) + g^1(X^\sigma, \nabla_{\partial/\partial \dot{x}^k}^\sigma X_i^\sigma), \\ \frac{\partial}{\partial \dot{x}^k}(g^1(X^\sigma, \partial/\partial \dot{x}^j)) &= g^1(\nabla_{\partial/\partial \dot{x}^k}^\sigma X^\sigma, \partial/\partial \dot{x}^j) + g^1(X^\sigma, \nabla_{\partial/\partial \dot{x}^k}^\sigma \partial/\partial \dot{x}^j), \\ \frac{\partial}{\partial \dot{x}^k}(g^1(X_i^\sigma, \partial/\partial \dot{x}^j)) &= g^1(\nabla_{\partial/\partial \dot{x}^k}^\sigma X_i^\sigma, \partial/\partial \dot{x}^j) + g^1(X_i^\sigma, \nabla_{\partial/\partial \dot{x}^k}^\sigma \partial/\partial \dot{x}^j), \end{aligned}$$

and the rest of conditions for $\nabla^\sigma g^1 = 0$ leads us to the following equations for $i, j, k = 1, \dots, n$:

$$\begin{aligned} X^\sigma(g^1(X_i^\sigma, X_j^\sigma)) &= -\frac{1}{2} \left(g^1(X_j^\sigma, X_k^\sigma) \frac{\partial F^k}{\partial \dot{x}^i} + g^1(X_i^\sigma, X_k^\sigma) \frac{\partial F^k}{\partial \dot{x}^j} \right), \\ X_k^\sigma(g^1(X_i^\sigma, X_j^\sigma)) &= -\frac{1}{2} \left(g^1(X_j^\sigma, X_h^\sigma) \frac{\partial^2 F^h}{\partial \dot{x}^i \partial \dot{x}^k} + g^1(X_i^\sigma, X_h^\sigma) \frac{\partial^2 F^h}{\partial \dot{x}^j \partial \dot{x}^k} \right), \\ \frac{\partial}{\partial \dot{x}^k}(g^1(X_i^\sigma, X_j^\sigma)) &= 0, \end{aligned}$$

$$\begin{aligned}
X^\sigma \left(g^1 \left(\frac{\partial}{\partial \dot{x}^i}, \frac{\partial}{\partial \dot{x}^j} \right) \right) &= -\frac{1}{2} \left(g^1 \left(\frac{\partial}{\partial \dot{x}^j}, \frac{\partial}{\partial \dot{x}^k} \right) \frac{\partial F^k}{\partial \dot{x}^i} + g^1 \left(\frac{\partial}{\partial \dot{x}^i}, \frac{\partial}{\partial \dot{x}^k} \right) \frac{\partial F^k}{\partial \dot{x}^j} \right), \\
X_k^\sigma \left(g^1 \left(\frac{\partial}{\partial \dot{x}^i}, \frac{\partial}{\partial \dot{x}^j} \right) \right) &= -\frac{1}{2} \left(g^1 \left(\frac{\partial}{\partial \dot{x}^j}, \frac{\partial}{\partial \dot{x}^h} \right) \frac{\partial^2 F^h}{\partial \dot{x}^i \partial \dot{x}^k} + g^1 \left(\frac{\partial}{\partial \dot{x}^i}, \frac{\partial}{\partial \dot{x}^h} \right) \frac{\partial^2 F^h}{\partial \dot{x}^j \partial \dot{x}^k} \right), \\
\frac{\partial}{\partial \dot{x}^k} \left(g^1 \left(\frac{\partial}{\partial \dot{x}^i}, \frac{\partial}{\partial \dot{x}^j} \right) \right) &= 0.
\end{aligned}$$

As the first group of three equations above is equivalent to the second group, taking (49) into account, these six equations reduce to the following:

$$\begin{aligned}
0 &= \delta_{ba} \left\{ 2X^\sigma \left((\Lambda^{-1})_i^b (\Lambda^{-1})_j^a \right) + \left((\Lambda^{-1})_j^b \frac{\partial F^k}{\partial \dot{x}^i} + (\Lambda^{-1})_i^b \frac{\partial F^k}{\partial \dot{x}^j} \right) (\Lambda^{-1})_k^a \right\}, \\
0 &= \delta_{ba} \left\{ 2X_k^\sigma \left((\Lambda^{-1})_i^b (\Lambda^{-1})_j^a \right) \right. \\
&\quad \left. + \left((\Lambda^{-1})_j^b \frac{\partial^2 F^h}{\partial \dot{x}^i \partial \dot{x}^k} + (\Lambda^{-1})_i^b \frac{\partial^2 F^h}{\partial \dot{x}^j \partial \dot{x}^k} \right) (\Lambda^{-1})_h^a \right\}, \\
&\quad \delta_{ba} \frac{\partial}{\partial \dot{x}^k} \left((\Lambda^{-1})_i^b (\Lambda^{-1})_j^a \right) = 0.
\end{aligned}$$

For $a \neq b$ all these equations vanish identically and for $a = b$, by writing $U = (\Lambda^t)^{-1} \Lambda^{-1}$, such equations are

$$(50) \quad X^\sigma(U) + UW + (UW)^t = 0,$$

$$(51) \quad X_k^\sigma(U) + UV_k + (UV_k)^t = 0,$$

where $V_k = (v_{ik}^h)$, $v_{ik}^h = \frac{1}{2} \frac{\partial^2 F^h}{\partial \dot{x}^i \partial \dot{x}^k}$, together with the following:

$$(52) \quad \frac{\partial U}{\partial \dot{x}^k} = 0.$$

The equations (52) are equivalent to saying the entries u_j^i of U belong to $C^\infty(M^0)$. Hence, the equations (50), (51) can also be rewritten respectively as

$$\begin{aligned}
\frac{\partial U}{\partial t} + \dot{x}^i \frac{\partial U}{\partial x^i} + UW + (UW)^t &= 0, \\
\frac{\partial U}{\partial x^k} + UV_k + (UV_k)^t &= 0.
\end{aligned}$$

Finally, we claim that the second equation is a consequence of the first one and (52), as follows taking derivatives with respect to \dot{x}^k in the first equation above. \square

Remark 6.4. As a simple—but rather long—computations shows, the integrability conditions for the system (50), (51), and (52) are

$$\begin{aligned}
(53) \quad 0 &= UR_{ij} + (R_{ij})^t U, \quad 1 \leq i \leq j \leq n, \\
0 &= UB_{ij} + (B_{ij})^t U, \quad 1 \leq i < j \leq n, \\
0 &= UA_j + (A_j)^t U, \quad 1 \leq j \leq n,
\end{aligned}$$

where the square matrices A_j , B_{ij} , R_{ij} are given by $A_j = (A_{kj}^h)$, $B_{ij} = (B_{ijk}^h)$, $R_{ij} = (R_{ijk}^h)$, and the functions A_{kj}^h , B_{ijk}^h , R_{ijk}^h are defined by the formulas (34), (35), (36), respectively. If U is a positive definite symmetric matrix, then

$$\mathfrak{g}_U = \{X \in \mathfrak{gl}(n, \mathbb{R}) : UX + X^t U = 0\}$$

is a Lie subalgebra of dimension $\frac{1}{2}n(n-1)$. Hence, if a matrix U exists satisfying (53), then the dimension of the subalgebra in $\mathfrak{gl}(n, \mathbb{R})$ generated by the $n(n+1)$ matrices $\{A_h\}_{h=1}^n$, $\{B_{ij}\}_{1 \leq i < j \leq n}$, $\{R_{kl}\}_{1 \leq k \leq l \leq n}$ must be $\leq \frac{1}{2}n(n-1)$.

Remark 6.5. The matrix U depends only on the symmetric part of the polar decomposition of Λ ; namely, if $\Lambda = SR$, where $R \in SO(n)$ and S is a positive definite symmetric matrix, then $U = S^{-2}$.

Example 6.6. According to Remark 5.4, the Levi-Civita connection of a pseudo-Riemannian metric $g = g_{ij}dx^i \otimes dx^j$ on M induces a homogeneous quadratic SODE σ independent of t , given by $F^h = F_{ij}^h \dot{x}^i \dot{x}^j$, where

$$F_{ij}^h = -\frac{1}{2}g^{hk} \left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k} \right).$$

As in Remark 5.4 the component of the Chern connection in the tangent bundle TM coincides with the horizontal lift ∇^H of the Levi-Civita connection ∇ of g , which is known to be a metric connection with respect to the metric g_H on TM defined in [28, p. 137]; in the present case, g_H takes the form $g_H = g_{ij}(\omega^i \otimes \varpi^j + \varpi^i \otimes \omega^j) + g_{ij}\dot{x}^i(dt \otimes \varpi^j + \varpi^j \otimes dt)$. Using this fact, it is readily seen that ∇^σ parallelizes the metric $h^1 = dt \otimes dt + g_H$ on M^1 .

Moreover, the equations (48) hold identically for $U = (g_{ij})_{i,j=1}^n$, and using Remark 6.5 without loss of generality we can assume $\Lambda = U^{-\frac{1}{2}}$. From the results of Proposition 6.3 we thus conclude that ∇^σ parallelizes also the metric $g^1 = dt \otimes dt + g_{ij}\omega^i \otimes \omega^j + g_{ij}\varpi^i \otimes \varpi^j$. If g is a Riemannian metric, then g^1 is also Riemannian but h^1 is maximally hyperbolic; in fact, its signature is $(n+1, n)$.

7 Naturality of the Chern connection

A diffeomorphism $\Phi: M^0 \rightarrow M^0$ is said to be a p -vertical automorphism of the submersion $p: M^0 \rightarrow \mathbb{R}$ if it takes the form $\Phi(t, x) = (t, \phi(t, x))$, $\forall (t, x) \in M^0$, $\phi \in C^\infty(M^0, M)$. The set of such transformations is a group with respect to composition of maps, denoted by $\text{Aut}^v(p)$. For each $r \geq 0$, every $\Phi \in \text{Aut}^v(p)$ induces a diffeomorphism $\Phi^{(r)}: M^r \rightarrow M^r$ by setting

$$(54) \quad \Phi^{(r)}(j_t^r \gamma) = j_t^r (\Phi \circ j^0 \gamma), \quad \forall \gamma \in C^\infty(\mathbb{R}, M).$$

If $f: N \rightarrow N$ is a diffeomorphism and $X \in \mathfrak{X}(N)$, then $f \cdot X \in \mathfrak{X}(N)$ is defined by $(f \cdot X)_X = f_*(X_{f^{-1}(x)})$, $\forall x \in N$.

The connection ∇^σ enjoys the important property of being functorial with respect to the p -vertical automorphisms; more precisely,

Theorem 7.1. For every SODE σ on M and every $\Phi \in \text{Aut}^v(p)$, let $\Phi \cdot \nabla^\sigma$ be the linear connection defined by,

$$(\Phi \cdot \nabla^\sigma)_X Y = \Phi^{(1)} \cdot \left((\nabla^\sigma)_{(\Phi^{(1)})^{-1} \cdot X} \left((\Phi^{(1)})^{-1} \cdot Y \right) \right), \quad \forall X, Y \in \mathfrak{X}(M^1),$$

and let $\Phi \cdot \sigma$ be the SODE defined as follows:

$$\begin{array}{ccc} \Phi \cdot \sigma & = & \Phi^{(2)} \circ \sigma \circ (\Phi^{(1)})^{-1}, \\ M^1 & \xrightarrow{\sigma} & M^2 \\ (\Phi^{(1)})^{-1} \uparrow & & \downarrow \Phi^{(2)} \\ M^1 & \xrightarrow{\Phi \cdot \sigma} & M^2 \end{array}$$

Then,

$$\Phi \cdot \nabla^\sigma = \nabla^{\Phi \cdot \sigma}.$$

Proof. The definition of $\Phi \cdot \sigma$ makes sense, as it is readily seen that the map $\Phi \cdot \sigma: M^1 \rightarrow M^2$ is a section of p^{21} .

By applying the chain rule twice, we obtain

$$(55) \quad \dot{x}^h \circ \Phi^{(1)} = \phi_t^h + \phi_i^h \dot{x}^i,$$

$$(56) \quad \ddot{x}^h \circ \Phi^{(2)} = \phi_{tt}^h + 2\phi_{ti}^h \dot{x}^i + \phi_{ij}^h \dot{x}^i \dot{x}^j + \phi_i^h \ddot{x}^i,$$

where $\phi^h = x^h \circ \Phi$, $\phi_t^h = \frac{\partial \phi^h}{\partial t}$, $\phi_i^h = \frac{\partial \phi^h}{\partial x^i}$, $\phi_{ti}^h = \frac{\partial^2 \phi^h}{\partial t \partial x^i}$, $\phi_{ij}^h = \frac{\partial^2 \phi^h}{\partial x^i \partial x^j}$, etc. Hence

$$(57) \quad \Phi^{(1)} \cdot \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + (\phi_t^h \circ \Phi^{-1}) \frac{\partial}{\partial x^h} + (\phi_{ti}^h + \phi_{ti}^h \dot{x}^i) \circ (\Phi^{(1)})^{-1} \frac{\partial}{\partial \dot{x}^h},$$

$$(58) \quad \Phi^{(1)} \cdot \frac{\partial}{\partial x^i} = (\phi_i^h \circ \Phi^{-1}) \frac{\partial}{\partial x^h} + (\phi_{ti}^h + \phi_{ij}^h \dot{x}^j) \circ (\Phi^{(1)})^{-1} \frac{\partial}{\partial \dot{x}^h},$$

$$(59) \quad \Phi^{(1)} \cdot \frac{\partial}{\partial \dot{x}^i} = (\phi_i^h \circ \Phi^{-1}) \frac{\partial}{\partial \dot{x}^h}.$$

We first prove that for every $\Phi \in \text{Aut}^v(p)$ the following formula holds:

$$(60) \quad \Phi^{(1)} \cdot X^\sigma = X^{\Phi \cdot \sigma}.$$

If $F^h(\Phi \cdot \sigma) = \dot{x}^h \circ (\Phi \cdot \sigma)$, then from the formula (56) we obtain

$$(61) \quad F^h(\Phi \cdot \sigma) \circ \Phi^{(1)} = \phi_{tt}^h + 2\phi_{ti}^h \dot{x}^i + \phi_{ij}^h \dot{x}^i \dot{x}^j + \phi_i^h F^i(\sigma),$$

and by using the formulas (55), (57), (58), and (59) we deduce

$$\begin{aligned} \Phi^{(1)} \cdot X^\sigma &= \frac{\partial}{\partial t} + (\phi_t^h \circ \Phi^{-1}) \frac{\partial}{\partial x^h} + (\phi_{ti}^h + \phi_{ti}^h \dot{x}^i) \circ (\Phi^{(1)})^{-1} \frac{\partial}{\partial \dot{x}^h} \\ &\quad + \left(\dot{x}^i \circ (\Phi^{(1)})^{-1} \right) \left\{ (\phi_i^h \circ \Phi^{-1}) \frac{\partial}{\partial x^h} + (\phi_{ti}^h + \phi_{ij}^h \dot{x}^j) \circ (\Phi^{(1)})^{-1} \frac{\partial}{\partial \dot{x}^h} \right\} \\ &\quad + \left(F^i(\sigma) \circ (\Phi^{(1)})^{-1} \right) (\phi_i^h \circ \Phi^{-1}) \frac{\partial}{\partial \dot{x}^h} \\ &= X^{\Phi \cdot \sigma}. \end{aligned}$$

Similarly, we obtain

$$(62) \quad \Phi^{(1)} \cdot X_i^\sigma = (\phi_i^h \circ \Phi^{-1}) X_h^{\Phi \cdot \sigma},$$

$$(63) \quad \Phi^{(1)} \cdot \frac{\partial}{\partial \dot{x}^i} = (\phi_i^h \circ \Phi^{-1}) \frac{\partial}{\partial \dot{x}^h}.$$

According to (23) we know the connection $\nabla^{\Phi \cdot \sigma}$ is given by,

$$(64) \quad \begin{aligned} \nabla_{X^{\Phi \cdot \sigma}}^{\Phi \cdot \sigma} X^{\Phi \cdot \sigma} &= 0, \quad \nabla_{X^{\Phi \cdot \sigma}}^{\Phi \cdot \sigma} X_i^{\Phi \cdot \sigma} = -\frac{1}{2} \frac{\partial F^j(\Phi \cdot \sigma)}{\partial \dot{x}^i} X_j^{\Phi \cdot \sigma}, \quad \nabla_{X^{\Phi \cdot \sigma}}^{\Phi \cdot \sigma} \frac{\partial}{\partial \dot{x}^i} = -\frac{1}{2} \frac{\partial F^j(\Phi \cdot \sigma)}{\partial \dot{x}^i} \frac{\partial}{\partial \dot{x}^j}, \\ \nabla_{X_i^{\Phi \cdot \sigma}}^{\Phi \cdot \sigma} X^{\Phi \cdot \sigma} &= 0, \quad \nabla_{X_j^{\Phi \cdot \sigma}}^{\Phi \cdot \sigma} X_i^{\Phi \cdot \sigma} = -\frac{1}{2} \frac{\partial^2 F^k(\Phi \cdot \sigma)}{\partial \dot{x}^i \partial \dot{x}^j} X_k^{\Phi \cdot \sigma}, \quad \nabla_{X_i^{\Phi \cdot \sigma}}^{\Phi \cdot \sigma} \frac{\partial}{\partial \dot{x}^j} = -\frac{1}{2} \frac{\partial^2 F^k(\Phi \cdot \sigma)}{\partial \dot{x}^i \partial \dot{x}^j} \frac{\partial}{\partial \dot{x}^k}, \\ \nabla_{\frac{\partial}{\partial \dot{x}^i}}^{\Phi \cdot \sigma} X^{\Phi \cdot \sigma} &= 0, \quad \nabla_{\frac{\partial}{\partial \dot{x}^i}}^{\Phi \cdot \sigma} X_j^{\Phi \cdot \sigma} = 0, \quad \nabla_{\frac{\partial}{\partial \dot{x}^i}}^{\Phi \cdot \sigma} \frac{\partial}{\partial \dot{x}^j} = 0. \end{aligned}$$

Hence, we need only to prove that the formulas in (64) also hold when $\nabla^{\Phi \cdot \sigma}$ is replaced by $\Phi \cdot \nabla^\sigma$. We have

$$\begin{aligned} (\Phi \cdot \nabla^\sigma)_Y X^{\Phi \cdot \sigma} &= \Phi^{(1)} \cdot \left((\nabla^\sigma)_{(\Phi^{(1)})^{-1} \cdot Y} \left((\Phi^{(1)})^{-1} \cdot X^{\Phi \cdot \sigma} \right) \right) \\ &\stackrel{(60)}{=} \Phi^{(1)} \cdot \left(\nabla_{(\Phi^{(1)})^{-1} \cdot Y}^\sigma X^\sigma \right) \\ &= 0, \end{aligned}$$

for every $Y \in \mathfrak{X}(M^1)$. As Φ is a diffeomorphism, the matrix $(\phi_i^h)_{h,i=1,\dots,n}$ is non-singular; we set $\Psi = (\psi_i^h) = (\phi_i^h)^{-1}$. By replacing Φ (resp. σ) by Φ^{-1} (resp. $\Phi \cdot \sigma$) in (62) we obtain $(\Phi^{(1)})^{-1} \cdot X_i^{\Phi \cdot \sigma} = \psi_i^h X_h^\sigma$. Hence

$$\begin{aligned} (\Phi \cdot \nabla^\sigma)_{X^{\Phi \cdot \sigma}} X_i^{\Phi \cdot \sigma} &= \Phi^{(1)} \cdot \left((\nabla^\sigma)_{(\Phi^{(1)})^{-1} \cdot X^{\Phi \cdot \sigma}} \left((\Phi^{(1)})^{-1} \cdot X_i^{\Phi \cdot \sigma} \right) \right) \\ &= \Phi^{(1)} \cdot \left(\nabla_{X^\sigma}^\sigma (\psi_i^h X_h^\sigma) \right) \\ &= \Phi^{(1)} \cdot \left\{ \left(X^\sigma(\psi_i^h) - \frac{1}{2} \psi_i^j \frac{\partial F^h(\sigma)}{\partial \dot{x}^j} \right) X_h^\sigma \right\} \\ &= -\Phi^{(1)} \cdot \left\{ \left(\psi_r^h X^\sigma(\phi_j^r) + \frac{1}{2} \frac{\partial F^h(\sigma)}{\partial \dot{x}^j} \right) \psi_i^j X_h^\sigma \right\} \\ &= -\left\{ \left(\psi_r^h X^\sigma(\phi_j^r) + \frac{1}{2} \frac{\partial F^h(\sigma)}{\partial \dot{x}^j} \right) \psi_i^j \right\} \circ (\Phi^{(1)})^{-1} (\Phi^{(1)} \cdot X_h^\sigma) \\ \text{by virtue of (62)} &= -\left(X^\sigma(\phi_j^a) \psi_i^j \right) \circ (\Phi^{(1)})^{-1} X_a^{\Phi \cdot \sigma} \\ &\quad - \frac{1}{2} (\phi_h^a \circ \Phi^{-1}) \left(\frac{\partial F^h(\sigma)}{\partial \dot{x}^j} \psi_i^j \right) \circ (\Phi^{(1)})^{-1} X_a^{\Phi \cdot \sigma} \\ &= -\frac{1}{2} \frac{\partial F^a(\Phi \cdot \sigma)}{\partial \dot{x}^i} X_a^{\Phi \cdot \sigma}, \end{aligned}$$

as

$$(65) \quad \frac{\partial F^h(\Phi \cdot \sigma)}{\partial \dot{x}^i} = 2 \left(X^\sigma(\phi_b^h) \psi_i^b \right) \circ \Phi^{-1} + \left(\phi_a^h \frac{\partial F^a(\sigma)}{\partial \dot{x}^b} \psi_i^b \right) \circ (\Phi^{(1)})^{-1},$$

which follows taking derivatives with respect to \dot{x}^i in the formula (61) and taking (55) into account. Similarly,

$$\begin{aligned}
(\Phi \cdot \nabla^\sigma)_{X_j^{\Phi \cdot \sigma}} X_i^{\Phi \cdot \sigma} &= \Phi^{(1)} \cdot \left((\nabla^\sigma)_{(\Phi^{(1)})^{-1} \cdot X_j^{\Phi \cdot \sigma}} \left((\Phi^{(1)})^{-1} \cdot X_i^{\Phi \cdot \sigma} \right) \right) \\
&= \Phi^{(1)} \cdot \left(\nabla_{\psi_j^k X_k^\sigma}^\sigma (\psi_i^h X_h^\sigma) \right) \\
&= \Phi^{(1)} \cdot \left\{ \psi_j^k \left(X_k^\sigma (\psi_i^h) - \frac{1}{2} \psi_i^a \frac{\partial^2 F^h(\sigma)}{\partial \dot{x}^a \partial \dot{x}^k} \right) X_h^\sigma \right\} \\
&= \Phi^{(1)} \cdot \left\{ \psi_j^k \left(-\psi_r^h X_k^\sigma (\phi_a^r) - \frac{1}{2} \frac{\partial^2 F^h(\sigma)}{\partial \dot{x}^a \partial \dot{x}^k} \right) \psi_i^a X_h^\sigma \right\} \\
&= - \left\{ \psi_j^k \left(\psi_r^h X_k^\sigma (\phi_a^r) + \frac{1}{2} \frac{\partial^2 F^h(\sigma)}{\partial \dot{x}^a \partial \dot{x}^k} \right) \psi_i^a \right\} \circ (\Phi^{(1)})^{-1} (\Phi^{(1)} \cdot X_h^\sigma) \\
&= - \left\{ \psi_j^k \left(\frac{1}{2} \psi_r^h \phi_{ak}^r + \frac{1}{2} \frac{\partial^2 F^h(\sigma)}{\partial \dot{x}^a \partial \dot{x}^k} \right) \psi_i^a \right\} \circ (\Phi^{(1)})^{-1} (\phi_h^s \circ \Phi^{-1}) X_s^{\Phi \cdot \sigma} \\
&= -\frac{1}{2} \left\{ \phi_{ak}^s \psi_j^k \psi_i^a + \frac{\partial^2 F^h(\sigma)}{\partial \dot{x}^a \partial \dot{x}^k} \psi_j^k \psi_i^a \phi_h^s \right\} \circ (\Phi^{(1)})^{-1} X_s^{\Phi \cdot \sigma} \\
&= -\frac{1}{2} \frac{\partial^2 F^k(\Phi \cdot \sigma)}{\partial \dot{x}^i \partial \dot{x}^j} X_k^{\Phi \cdot \sigma},
\end{aligned}$$

as

$$(66) \quad \frac{\partial^2 F^h(\Phi \cdot \sigma)}{\partial \dot{x}^i \partial \dot{x}^j} = \left(\phi_{ab}^h \psi_i^a \psi_j^b + \phi_a^h \frac{\partial^2 F^a(\sigma)}{\partial \dot{x}^b \partial \dot{x}^c} \psi_j^c \psi_i^b \right) \circ (\Phi^{(1)})^{-1},$$

and also

$$\begin{aligned}
(\Phi \cdot \nabla^\sigma)_{\frac{\partial}{\partial \dot{x}^i}} X_j^{\Phi \cdot \sigma} &= \Phi^{(1)} \cdot \left((\nabla^\sigma)_{(\Phi^{(1)})^{-1} \cdot \frac{\partial}{\partial \dot{x}^i}} \left((\Phi^{(1)})^{-1} \cdot X_j^{\Phi \cdot \sigma} \right) \right) \\
&= \Phi^{(1)} \cdot \left(\nabla_{\psi_i^k \frac{\partial}{\partial \dot{x}^k}}^\sigma (\psi_j^h X_h^\sigma) \right) \\
&= \Phi^{(1)} \cdot \left(\psi_i^k \psi_j^h \nabla_{\frac{\partial}{\partial \dot{x}^k}}^\sigma X_h^\sigma \right) \\
&= 0.
\end{aligned}$$

Taking (65) into account, we obtain

$$\begin{aligned}
(\Phi \cdot \nabla^\sigma)_{X^{\Phi \cdot \sigma}} \frac{\partial}{\partial \dot{x}^i} &= \Phi^{(1)} \cdot \left((\nabla^\sigma)_{(\Phi^{(1)})^{-1} \cdot X^{\Phi \cdot \sigma}} \left((\Phi^{(1)})^{-1} \cdot \frac{\partial}{\partial \dot{x}^i} \right) \right) \\
&= \Phi^{(1)} \cdot \left(\nabla_{X^\sigma}^\sigma \left(\psi_i^h \frac{\partial}{\partial \dot{x}^h} \right) \right) \\
&= \Phi^{(1)} \cdot \left\{ \left(X^\sigma (\psi_i^h) - \frac{1}{2} \psi_i^j \frac{\partial F^h(\sigma)}{\partial \dot{x}^j} \right) \frac{\partial}{\partial \dot{x}^h} \right\} \\
&= -\Phi^{(1)} \cdot \left\{ \left(\psi_r^h X^\sigma (\phi_j^r) + \frac{1}{2} \frac{\partial F^h(\sigma)}{\partial \dot{x}^j} \right) \psi_i^j \frac{\partial}{\partial \dot{x}^h} \right\} \\
&= - \left\{ \left(\psi_r^h X^\sigma (\phi_j^r) + \frac{1}{2} \frac{\partial F^h(\sigma)}{\partial \dot{x}^j} \right) \psi_i^j \right\} \circ (\Phi^{(1)})^{-1} \left(\Phi^{(1)} \cdot \frac{\partial}{\partial \dot{x}^h} \right) \\
\text{by virtue of (63)} &= - \left\{ \left(X^\sigma (\phi_j^a) \psi_i^j \right) \circ (\Phi^{(1)})^{-1} \right\} \frac{\partial}{\partial \dot{x}^a} \\
&\quad - \frac{1}{2} \left\{ \left(\phi_h^a \psi_i^j \frac{\partial F^h(\sigma)}{\partial \dot{x}^j} \right) \circ (\Phi^{(1)})^{-1} \right\} \frac{\partial}{\partial \dot{x}^a} \\
&= -\frac{1}{2} \frac{\partial F^a(\Phi \cdot \sigma)}{\partial \dot{x}^i} \frac{\partial}{\partial \dot{x}^a}.
\end{aligned}$$

Similarly, taking (66) into account, we obtain

$$\begin{aligned}
(\Phi \cdot \nabla^\sigma)_{X_j^{\Phi \cdot \sigma}} \frac{\partial}{\partial \dot{x}^i} &= \Phi^{(1)} \cdot \left((\nabla^\sigma)_{(\Phi^{(1)})^{-1} \cdot X_j^{\Phi \cdot \sigma}} \left((\Phi^{(1)})^{-1} \cdot \frac{\partial}{\partial \dot{x}^i} \right) \right) \\
&= \Phi^{(1)} \cdot \left(\nabla_{\psi_j^k X_k^\sigma}^\sigma \left(\psi_i^h \frac{\partial}{\partial \dot{x}^h} \right) \right) \\
&= \Phi^{(1)} \cdot \left\{ \psi_j^k \left(\frac{\partial \psi_i^h}{\partial \dot{x}^k} - \frac{1}{2} \psi_i^a \frac{\partial^2 F^h(\sigma)}{\partial \dot{x}^a \partial \dot{x}^k} \right) \frac{\partial}{\partial \dot{x}^h} \right\} \\
&= \Phi^{(1)} \cdot \left\{ \psi_j^k \left(-\psi_r^h \frac{\partial \phi_a^r}{\partial \dot{x}^k} - \frac{1}{2} \frac{\partial^2 F^h(\sigma)}{\partial \dot{x}^a \partial \dot{x}^k} \right) \psi_i^a \frac{\partial}{\partial \dot{x}^h} \right\} \\
&= - \left\{ \psi_j^k \left(\psi_r^h \frac{\partial \phi_a^r}{\partial \dot{x}^k} + \frac{1}{2} \frac{\partial^2 F^h(\sigma)}{\partial \dot{x}^a \partial \dot{x}^k} \right) \psi_i^a \right\} \circ (\Phi^{(1)})^{-1} \left(\Phi^{(1)} \cdot \frac{\partial}{\partial \dot{x}^h} \right) \\
&= - \frac{1}{2} \frac{\partial^2 F^k(\Phi \cdot \sigma)}{\partial \dot{x}^i \partial \dot{x}^j} \frac{\partial}{\partial \dot{x}^k}.
\end{aligned}$$

Finally,

$$\begin{aligned}
(\Phi \cdot \nabla^\sigma)_{\frac{\partial}{\partial \dot{x}^i}} \frac{\partial}{\partial \dot{x}^j} &= \Phi^{(1)} \cdot \left(\nabla_{\psi_i^k \frac{\partial}{\partial \dot{x}^k}}^\sigma \left(\psi_j^h \frac{\partial}{\partial \dot{x}^h} \right) \right) \\
&= \Phi^{(1)} \cdot \left(\psi_i^k \psi_j^h \nabla_{\frac{\partial}{\partial \dot{x}^k}}^\sigma \frac{\partial}{\partial \dot{x}^h} \right) \\
&= 0.
\end{aligned}$$

□

Corollary 7.2. *If $\Phi \in \text{Aut}^v(p)$ and $X, Y, Z \in \mathfrak{X}(M^1)$ are vector fields $\Phi^{(1)}$ -related to $X', Y', Z' \in \mathfrak{X}(M^1)$ respectively, then $T^\sigma(X, Y)$ (resp. $R^\sigma(X, Y)Z$) is $\Phi^{(1)}$ -related to $T^{\Phi \cdot \sigma}(X', Y')$ (resp. $R^{\Phi \cdot \sigma}(X', Y')Z'$).*

Proof. It follows from [17, VI, Proposition 1.2, (2), (3)].

□

8 Differential invariants

Let $(p^{21})^r: J^r(p^{21}) \rightarrow M^1$ be the r -jet bundle of the submersion $p^{21}: M^2 \rightarrow M^1$. According to (54), every $\Phi \in \text{Aut}^v(p)$ induces in particular diffeomorphisms $\Phi^{(r)}: M^r \rightarrow M^r$, $r = 1, 2$, such that $p^{21} \circ \Phi^{(2)} = \Phi^{(1)} \circ p^{21}$. Hence, for every $r \geq 0$, the pair $\Phi^{(2)}, \Phi^{(1)}$ induces a transformation $(\Phi^{(2)})^{(r)}: J^r(p^{21}) \rightarrow J^r(p^{21})$ given by $(\Phi^{(2)})^{(r)}(j_\xi^r \sigma) = j_{\Phi^{(1)}(\xi)}^r(\Phi^{(2)} \circ \sigma \circ (\Phi^{(1)})^{-1})$. Let $\mathcal{U} \subseteq J^r(p^{21})$ be an open subset invariant under all these transformations. A smooth function $\mathcal{I}: \mathcal{U} \rightarrow \mathbb{R}$ is said to be a differential invariant of order r with respect to the group $\text{Aut}^v(p)$ if $\mathcal{I} \circ (\Phi^{(2)})^{(r)} = \mathcal{I}$ for all $\Phi \in \text{Aut}^v(p)$. If we set $I(\sigma, \xi) = \mathcal{I}(j_\xi^r \sigma)$, $\xi \in M^1$, for a given SODE σ on M , then the invariance condition above reads as follows: $I(\Phi \cdot \sigma, \Phi^{(1)}(\xi)) = I(\sigma, \xi)$, $\forall \xi \in M^1$, $\forall \Phi \in \text{Aut}^v(p)$, thus leading one to the naive definition of an invariant, as being a function depending on the components of σ and its partial derivatives up to a certain order, which remains unchanged under arbitrary changes of coordinates.

If $\Phi_t \in \text{Aut}^v(p)$ is the flow of a p -vertical vector field $X \in \mathfrak{X}(M^0)$, then $\Phi_t^{(2)}$ is the flow of a p^2 -vertical vector field $X^{(2)} \in \mathfrak{X}(M^2)$ and $(\Phi_t^{(2)})^{(r)}$ is the

flow of a vector field $(X^{(2)})^{(r)}$ on $J^r(p^{21})$. Every differential invariant of order r is a first integral of the distribution $\mathcal{D}^{(r)}$ on $J^r(p^{21})$ spanned by all the jet prolongations $(X^{(2)})^{(r)}$ of p -vertical vector fields.

We claim that the only first integrals of the distributions $\mathcal{D}^{(0)}$ and $\mathcal{D}^{(1)}$ are $(p^2)^*C^\infty(\mathbb{R})$ and $((p^{21})^1)^*(p^1)^*C^\infty(\mathbb{R})$, respectively. In fact, from the general formulas of jet prolongation of vector fields (e.g., see [20], [25]), we obtain

$$\begin{aligned} X &= u^i \frac{\partial}{\partial x^i}, \quad u^i \in C^\infty(M^0), \\ X^{(2)} &= u^i \frac{\partial}{\partial x^i} + v^i \frac{\partial}{\partial \dot{x}^i} + w^i \frac{\partial}{\partial \ddot{x}^i}, \\ (67) \quad v^i &= \frac{\partial u^i}{\partial t} + \frac{\partial u^i}{\partial x^h} \dot{x}^h, \\ (68) \quad w^i &= \frac{\partial^2 u^i}{\partial t^2} + 2 \frac{\partial^2 u^i}{\partial t \partial x^h} \dot{x}^h + \frac{\partial^2 u^i}{\partial x^h \partial x^k} \dot{x}^h \dot{x}^k + \frac{\partial u^i}{\partial x^h} \ddot{x}^h. \end{aligned}$$

As the values of u^i , $\partial u^i / \partial t$, $\partial u^i / \partial x^h$, and $\partial^2 u^i / \partial t^2$ can arbitrarily be taken at a given point $j_t^2 \gamma \in M^2$, we conclude that the distribution on M^2 generated by all the vector fields $X^{(2)}$ span the subbundle of p^2 -vertical tangent vectors. Hence, the only differential invariants of order 0 are the functions in $(p^2)^*C^\infty(\mathbb{R})$.

By again computing the jet prolongation, we obtain

$$\begin{aligned} (X^{(2)})^{(1)} &= u^i \frac{\partial}{\partial x^i} + v^i \frac{\partial}{\partial \dot{x}^i} + w^i \frac{\partial}{\partial \ddot{x}^i} + w_t^i \frac{\partial}{\partial \ddot{x}_t^i} + w_a^i \frac{\partial}{\partial \ddot{x}_a^i} + w_{\dot{a}}^i \frac{\partial}{\partial \ddot{x}_{\dot{a}}^i}, \\ (69) \quad w_t^i &= \frac{\partial^3 u^i}{\partial t^3} + 2 \frac{\partial^3 u^i}{\partial t^2 \partial x^h} \dot{x}^h + \frac{\partial^3 u^i}{\partial t \partial x^h \partial x^k} \dot{x}^h \dot{x}^k + \frac{\partial^2 u^i}{\partial t \partial x^h} \ddot{x}^h \\ &\quad + \frac{\partial u^i}{\partial x^h} \ddot{x}_t^h - \frac{\partial u^a}{\partial t} \ddot{x}_a^i - \frac{\partial^2 u^a}{\partial t^2} \ddot{x}_a^i - \frac{\partial^2 u^a}{\partial t \partial x^h} \dot{x}^h \ddot{x}_a^i, \\ (70) \quad w_a^i &= \frac{\partial^3 u^i}{\partial t^2 \partial x^a} + 2 \frac{\partial^3 u^i}{\partial t \partial x^a \partial x^h} \dot{x}^h + \frac{\partial^3 u^i}{\partial x^a \partial x^h \partial x^k} \dot{x}^h \dot{x}^k + \frac{\partial^2 u^i}{\partial x^a \partial x^h} \ddot{x}^h \\ &\quad + \frac{\partial u^i}{\partial x^h} \ddot{x}_a^h - \frac{\partial u^b}{\partial x^a} \ddot{x}_b^i - \frac{\partial^2 u^b}{\partial t \partial x^a} \ddot{x}_b^i - \frac{\partial^2 u^b}{\partial x^a \partial x^h} \dot{x}^h \ddot{x}_b^i, \\ (70) \quad w_{\dot{a}}^i &= 2 \frac{\partial^2 u^i}{\partial t \partial x^a} + 2 \frac{\partial^2 u^i}{\partial x^a \partial x^h} \dot{x}^h + \frac{\partial u^i}{\partial x^h} \ddot{x}_a^h - \frac{\partial u^r}{\partial x^a} \ddot{x}_r^i. \end{aligned}$$

By collecting the derivatives of the functions u^i in the expression above for $(X^{(2)})^{(1)}$ we conclude

$$\begin{aligned} (X^{(2)})^{(1)} &= u^r \chi_{00}^r + \frac{\partial u^r}{\partial t} \chi_t^r + \frac{\partial u^r}{\partial x^a} \chi_a^r + \frac{\partial^2 u^r}{\partial t^2} \chi_{tt}^r + \frac{\partial^2 u^r}{\partial t \partial x^a} \chi_{ta}^r + \sum_{a \leq b} \frac{\partial^2 u^r}{\partial x^a \partial x^b} \chi_{a \leq b}^r \\ &\quad + \frac{\partial^3 u^r}{\partial t^3} \chi_{ttt}^r + \frac{\partial^3 u^r}{\partial t^2 \partial x^a} \chi_{tta}^r + \sum_{a \leq b} \frac{\partial^3 u^r}{\partial t \partial x^a \partial x^b} \chi_{t, a \leq b}^r + \sum_{a \leq b \leq c} \frac{\partial^3 u^r}{\partial x^a \partial x^b \partial x^c} \chi_{a \leq b \leq c}^r, \end{aligned}$$

where

$$\begin{aligned} \chi_{00}^r &= \frac{\partial}{\partial x^r}, \\ \chi_t^r &= \frac{\partial}{\partial \dot{x}^r} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_t^i}, \\ \chi_a^r &= \dot{x}^a \frac{\partial}{\partial \dot{x}^r} + \ddot{x}_a^i \frac{\partial}{\partial \dot{x}^r} + \ddot{x}_t^a \frac{\partial}{\partial \ddot{x}_t^i} + \ddot{x}_h^a \frac{\partial}{\partial \ddot{x}_h^i} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_a^i} - \ddot{x}_{\dot{r}}^i \frac{\partial}{\partial \ddot{x}_{\dot{a}}^i} + \ddot{x}_b^a \frac{\partial}{\partial \ddot{x}_b^i}, \\ \chi_{tt}^r &= \frac{\partial}{\partial \ddot{x}^r} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_t^i}, \\ \chi_{ta}^r &= \dot{x}^a \left(2 \frac{\partial}{\partial \dot{x}^r} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_t^i} \right) + \ddot{x}_a^i \frac{\partial}{\partial \ddot{x}_t^i} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_a^i} + 2 \frac{\partial}{\partial \ddot{x}_a^i}, \end{aligned}$$

$$\begin{aligned}\chi_{a \leq b}^r = & \frac{1}{1+\delta_{ab}} \left\{ 2\dot{x}^a \dot{x}^b \frac{\partial}{\partial \dot{x}^r} + \ddot{x}^b \frac{\partial}{\partial \dot{x}_a^r} + \ddot{x}^a \frac{\partial}{\partial \dot{x}_b^r} \dot{x}^b \ddot{x}_r^i \frac{\partial}{\partial \dot{x}_a^i} - \dot{x}^a \ddot{x}_r^i \frac{\partial}{\partial \dot{x}_b^i} \right. \\ & \left. + 2\dot{x}^b \frac{\partial}{\partial \dot{x}_a^r} + 2\dot{x}^a \frac{\partial}{\partial \dot{x}_b^r} \right\},\end{aligned}$$

$$\begin{aligned}\chi_{ttt}^r &= \frac{\partial}{\partial \dot{x}_t^r}, \\ \chi_{tta}^r &= 2\dot{x}^a \frac{\partial}{\partial \dot{x}_t^r} + \frac{\partial}{\partial \dot{x}_a^r}, \\ \chi_{t,a \leq b}^r &= \frac{2}{1+\delta_{ab}} \left\{ \dot{x}^a \dot{x}^b \frac{\partial}{\partial \dot{x}_t^r} + \dot{x}^b \frac{\partial}{\partial \dot{x}_a^r} + \dot{x}^a \frac{\partial}{\partial \dot{x}_b^r} \right\}, \\ \chi_{a \leq b \leq c}^r &= \frac{2}{(1+\delta_{ab}+\delta_{bc})!} \left\{ \dot{x}^b \dot{x}^c \frac{\partial}{\partial \dot{x}_a^r} + \dot{x}^a \dot{x}^c \frac{\partial}{\partial \dot{x}_b^r} + \dot{x}^a \dot{x}^b \frac{\partial}{\partial \dot{x}_c^r} \right\},\end{aligned}$$

and $(t, x^i, \dot{x}^i, \ddot{x}_t^i, \ddot{x}_a^i, \ddot{x}_b^i)$ is the induced coordinate system on $J^1(p^{21})$, namely

$$\ddot{x}_t^i(j_\xi^1 \sigma) = \frac{\partial F^i}{\partial t}(\xi), \quad \ddot{x}_a^i(j_\xi^1 \sigma) = \frac{\partial F^i}{\partial x^a}(\xi), \quad \ddot{x}_b^i(j_\xi^1 \sigma) = \frac{\partial F^i}{\partial \dot{x}^a}(\xi).$$

Accordingly, $\chi_{00}^r, \chi_t^r, \chi_a^r, \chi_{tt}^r, \chi_{ta}^r, \chi_{a \leq b}^r, \chi_{ttt}^r, \chi_{tta}^r, \chi_{ta \leq b}^r, \chi_{a \leq b \leq c}^r$ constitute a system of generators for the distribution $\mathcal{D}^{(1)}$. From the expressions above for $\chi_{00}^r, \chi_{ttt}^r, \chi_t^r, \chi_{tt}^r, \chi_{tta}^r, \chi_{ta}^r$ we obtain

$$\begin{aligned}\frac{\partial}{\partial x^r} &= \chi_{00}^r, & \frac{\partial}{\partial \dot{x}^r} &= \chi_t^r + \ddot{x}_r^i \chi_{ttt}^i, & \frac{\partial}{\partial \dot{x}_t^r} &= \chi_{ttt}^r, \\ \frac{\partial}{\partial \dot{x}^r} &= \chi_{tt}^r + \ddot{x}_r^i \chi_{ttt}^i, & \frac{\partial}{\partial \dot{x}_a^r} &= \chi_{tta}^r - 2\dot{x}^a \chi_{ttt}^r, \\ 2\frac{\partial}{\partial \dot{x}_a^r} &= \chi_{ta}^r - 2\dot{x}^a \chi_{tt}^r - 3\dot{x}^a \ddot{x}_r^i \chi_{ttt}^i - \ddot{x}^a \chi_{ttt}^r + \ddot{x}_r^i \chi_{tta}^i.\end{aligned}$$

As in the previous case, the first-order differential invariants are none other than the functions in $((p^{21})^1)^*(p^1)^*C^\infty(\mathbb{R})$.

Lemma 8.1. *The rank of $\mathcal{D}^{(2)}$ is 11 if $\dim M = 1$ and $\frac{1}{2}n(3n^2 + 11n + 10)$ if $\dim M = n > 1$.*

Proof. By computing the second jet prolongation, the following formulas are obtained:

$$\begin{aligned}(X^{(2)})^{(2)} &= u^i \frac{\partial}{\partial x^i} + v^i \frac{\partial}{\partial \dot{x}^i} + w^i \frac{\partial}{\partial \dot{x}_t^i} + w_t^i \frac{\partial}{\partial \dot{x}_t^i} + w_a^i \frac{\partial}{\partial \dot{x}_a^i} + w_{\dot{a}}^i \frac{\partial}{\partial \dot{x}_a^i} + w_{tt}^i \frac{\partial}{\partial \dot{x}_{tt}^i} \\ &\quad + w_{ta}^i \frac{\partial}{\partial \dot{x}_{ta}^i} + w_{t\dot{a}}^i \frac{\partial}{\partial \dot{x}_{ta}^i} + w_{a \leq b}^i \frac{\partial}{\partial \dot{x}_{a \leq b}^i} + w_{ab}^i \frac{\partial}{\partial \dot{x}_{ab}^i} + w_{a \leq \dot{b}}^i \frac{\partial}{\partial \dot{x}_{a \leq \dot{b}}^i}, \\ w_{tt}^i &= \frac{\partial^4 u^i}{\partial t^4} + 2\frac{\partial^4 u^i}{\partial t^3 \partial x^h} \dot{x}^h + \frac{\partial^4 u^i}{\partial t^2 \partial x^h \partial x^k} \dot{x}^h \dot{x}^k - \frac{\partial^3 u^r}{\partial t^3} \ddot{x}_r^i + \frac{\partial^3 u^i}{\partial t^2 \partial x^h} \ddot{x}^h - \frac{\partial^3 u^r}{\partial t^2 \partial x^h} \dot{x}^h \ddot{x}_r^i \\ &\quad - \frac{\partial^2 u^r}{\partial t^2} \ddot{x}_r^i - 2\frac{\partial^2 u^r}{\partial t^2} \ddot{x}_{tr}^i + 2\frac{\partial^2 u^i}{\partial t \partial x^h} \ddot{x}_t^h - 2\frac{\partial^2 u^r}{\partial t \partial x^h} \dot{x}^h \ddot{x}_{tr}^i - 2\frac{\partial u^r}{\partial t} \ddot{x}_{tr}^i + \frac{\partial u^i}{\partial x^h} \ddot{x}_{tt}^h, \\ w_{ta}^i &= \frac{\partial^4 u^i}{\partial t^3 \partial x^a} + 2\frac{\partial^4 u^i}{\partial t^2 \partial x^a \partial x^h} \dot{x}^h + \frac{\partial^4 u^i}{\partial t \partial x^h \partial x^k \partial x^a} \dot{x}^h \dot{x}^k - \frac{\partial^3 u^r}{\partial t^2 \partial x^a} \ddot{x}_r^i + \frac{\partial^3 u^i}{\partial t \partial x^a \partial x^h} \ddot{x}^h \\ &\quad - \frac{\partial^3 u^r}{\partial t \partial x^h \partial x^a} \dot{x}^h \ddot{x}_r^i - \frac{\partial^2 u^r}{\partial t^2} \ddot{x}_{ar}^i - \frac{\partial^2 u^r}{\partial t \partial x^a} \ddot{x}_r^i + \frac{\partial^2 u^i}{\partial t \partial x^h} \ddot{x}_a^h - \frac{\partial^2 u^r}{\partial t \partial x^a} \ddot{x}_{tr}^i \\ &\quad - \frac{\partial^2 u^r}{\partial t \partial x^h} \dot{x}^h \ddot{x}_{ar}^i + \frac{\partial^2 u^i}{\partial x^h \partial x^a} \ddot{x}_t^h - \frac{\partial^2 u^r}{\partial x^a \partial x^h} \dot{x}^h \ddot{x}_{tr}^i - \frac{\partial u^r}{\partial t} \ddot{x}_{ar}^i - \frac{\partial u^r}{\partial x^a} \ddot{x}_{tr}^i + \frac{\partial u^i}{\partial x^h} \ddot{x}_{ta}^h,\end{aligned}$$

$$(71) \quad w_{ta}^i = 2 \frac{\partial^3 u^i}{\partial t^2 \partial x^a} + 2 \frac{\partial^3 u^i}{\partial t \partial x^a \partial x^h} \dot{x}^h - \frac{\partial^2 u^r}{\partial t^2} \ddot{x}_{ar}^i + \frac{\partial^2 u^i}{\partial t \partial x^h} \dot{x}_a^h - \frac{\partial^2 u^r}{\partial t \partial x^a} \ddot{x}_r^i \\ - \frac{\partial^2 u^r}{\partial t \partial x^h} \dot{x}^h \ddot{x}_{ar}^i - \frac{\partial u^r}{\partial t} \ddot{x}_{ar}^i + \frac{\partial u^i}{\partial x^h} \ddot{x}_{ta}^h - \frac{\partial u^r}{\partial x^a} \ddot{x}_{tr}^i,$$

$$w_{a \leq b}^i = \frac{\partial^4 u^i}{\partial t^2 \partial x^a \partial x^b} + 2 \frac{\partial^4 u^i}{\partial t \partial x^a \partial x^h \partial x^b} \dot{x}^h + \frac{\partial^4 u^i}{\partial x^a \partial x^b \partial x^h \partial x^k} \dot{x}^h \dot{x}^k \frac{\partial^3 u^r}{\partial t \partial x^a \partial x^b} \ddot{x}_r^i \\ + \frac{\partial^3 u^i}{\partial x^a \partial x^b \partial x^h} \ddot{x}^h - \frac{\partial^3 u^r}{\partial x^a \partial x^b \partial x^h} \dot{x}^h \ddot{x}_r^i - \frac{\partial^2 u^r}{\partial t \partial x^a} \ddot{x}_{rb}^i - \frac{\partial^2 u^r}{\partial t \partial x^b} \ddot{x}_{ar}^i \\ + \frac{\partial^2 u^i}{\partial x^h \partial x^b} \ddot{x}_a^h + \frac{\partial^2 u^i}{\partial x^a \partial x^h} \ddot{x}_b^h - \frac{\partial^2 u^r}{\partial x^a \partial x^b} \ddot{x}_r^i - \frac{\partial^2 u^r}{\partial x^a \partial x^h} \dot{x}^h \ddot{x}_{rb}^i \\ - \frac{\partial^2 u^r}{\partial x^b \partial x^h} \dot{x}^h \ddot{x}_{ar}^i - \frac{\partial u^r}{\partial x^b} \ddot{x}_{ar}^i - \frac{\partial u^r}{\partial x^a} \ddot{x}_{rb}^i + \frac{\partial u^i}{\partial x^h} \ddot{x}_{a \leq b}^h,$$

$$(72) \quad w_{ab}^i = 2 \frac{\partial^3 u^i}{\partial t \partial x^a \partial x^b} + 2 \frac{\partial^3 u^i}{\partial x^a \partial x^b \partial x^k} \dot{x}^k - \frac{\partial^2 u^r}{\partial t \partial x^a} \ddot{x}_{rb}^i - \frac{\partial^2 u^r}{\partial x^a \partial x^b} \ddot{x}_r^i + \frac{\partial^2 u^i}{\partial x^a \partial x^h} \ddot{x}_b^h \\ - \frac{\partial^2 u^r}{\partial x^a \partial x^h} \dot{x}^h \ddot{x}_{rb}^i + \frac{\partial u^i}{\partial x^h} \ddot{x}_{ab}^h - \frac{\partial u^r}{\partial x^a} \ddot{x}_{rb}^i - \frac{\partial u^r}{\partial x^b} \ddot{x}_{ar}^i,$$

$$(73) \quad w_{a \leq b}^i = 2 \frac{\partial^2 u^i}{\partial x^a \partial x^b} + \frac{\partial u^i}{\partial x^h} \ddot{x}_{a \leq b}^h - \frac{\partial u^r}{\partial x^a} \ddot{x}_{br}^i - \frac{\partial u^r}{\partial x^b} \ddot{x}_{ar}^i,$$

where we assume $\ddot{x}_{ba}^i = \ddot{x}_{a \leq b}^i$ for $a \leq b$ and $(t, x^i, \dot{x}^i, \ddot{x}_t^i, \ddot{x}_a^i, \ddot{x}_{\dot{a}}^i, \ddot{x}_{tt}^i, \ddot{x}_{ta}^i, \ddot{x}_{ta}^i, \ddot{x}_{a \leq b}^i, \ddot{x}_{ab}^i, \ddot{x}_{a \leq b}^i)$ is the induced coordinate system on $J^2(p^{21})$. Hence

$$(X^{(2)})^{(2)} = u^r \mathcal{K}_{00}^r + \frac{\partial u^r}{\partial t} \mathcal{K}_t^r + \frac{\partial u^r}{\partial x^a} \mathcal{K}_a^r + \frac{\partial^2 u^r}{\partial t^2} \mathcal{K}_{tt}^r + \frac{\partial^2 u^r}{\partial t \partial x^a} \mathcal{K}_{ta}^r + \sum_{a \leq b} \frac{\partial^2 u^r}{\partial x^a \partial x^b} \mathcal{K}_{a \leq b}^r \\ + \frac{\partial^3 u^r}{\partial t^3} \mathcal{K}_{ttt}^r + \frac{\partial^3 u^r}{\partial t^2 \partial x^a} \mathcal{K}_{tta}^r + \sum_{a \leq b} \frac{\partial^3 u^r}{\partial t \partial x^a \partial x^b} \mathcal{K}_{t, a \leq b}^r \\ + \sum_{a \leq b \leq c} \frac{\partial^3 u^r}{\partial x^a \partial x^b \partial x^c} \mathcal{K}_{a \leq b \leq c}^r + \frac{\partial^4 u^r}{\partial t^4} \mathcal{K}_{tttt}^r + \frac{\partial^4 u^r}{\partial t^3 \partial x^a} \mathcal{K}_{ttta}^r + \sum_{a \leq b} \frac{\partial^4 u^r}{\partial t^2 \partial x^a \partial x^b} \mathcal{K}_{tt, a \leq b}^r \\ + \sum_{a \leq b \leq c} \frac{\partial^4 u^r}{\partial t \partial x^a \partial x^b \partial x^c} \mathcal{K}_{t, a \leq b \leq c}^r + \sum_{a \leq b \leq c \leq d} \frac{\partial^4 u^r}{\partial x^a \partial x^b \partial x^c \partial x^d} \mathcal{K}_{a \leq b \leq c \leq d}^r,$$

where

$$\mathcal{K}_{00}^r = \frac{\partial}{\partial x^r}, \\ \mathcal{K}_t^r = \frac{\partial}{\partial x^r} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_t^i} - 2 \ddot{x}_{tr}^i \frac{\partial}{\partial \ddot{x}_{tt}^i} - \ddot{x}_{ar}^i \frac{\partial}{\partial \ddot{x}_{ta}^i} - \ddot{x}_{ra}^i \frac{\partial}{\partial \ddot{x}_{ta}^i}, \\ \mathcal{K}_a^r = \dot{x}^a \frac{\partial}{\partial x^r} + \ddot{x}_a^a \frac{\partial}{\partial x^r} + \ddot{x}_t^a \frac{\partial}{\partial \ddot{x}_t^r} + \ddot{x}_b^a \frac{\partial}{\partial \ddot{x}_b^r} + \ddot{x}_b^a \frac{\partial}{\partial \ddot{x}_b^r} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_a^i} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_a^i} + \ddot{x}_{tt}^a \frac{\partial}{\partial \ddot{x}_{tt}^r} \\ - \ddot{x}_{tr}^i \frac{\partial}{\partial \ddot{x}_{ta}^i} + \ddot{x}_{tb}^a \frac{\partial}{\partial \ddot{x}_{tb}^r} + \ddot{x}_{tb}^a \frac{\partial}{\partial \ddot{x}_{tb}^r} - \ddot{x}_{tr}^i \frac{\partial}{\partial \ddot{x}_{ta}^i} - \sum_{c \leq a} \ddot{x}_{cr}^i \frac{\partial}{\partial \ddot{x}_{c \leq a}^i} - \sum_{a \leq b} \ddot{x}_{rb}^i \frac{\partial}{\partial \ddot{x}_{a \leq b}^i} \\ + \sum_{c \leq b} \ddot{x}_{cb}^a \frac{\partial}{\partial \ddot{x}_{c \leq b}^r} + \ddot{x}_{cb}^a \frac{\partial}{\partial \ddot{x}_{cb}^r} - \ddot{x}_{rb}^i \frac{\partial}{\partial \ddot{x}_{ab}^i} - \ddot{x}_{cr}^i \frac{\partial}{\partial \ddot{x}_{ca}^i} + \sum_{c \leq b} \ddot{x}_{c \leq b}^a \frac{\partial}{\partial \ddot{x}_{c \leq b}^r} \\ - \sum_{a \leq b} \ddot{x}_{br}^i \frac{\partial}{\partial \ddot{x}_{a \leq b}^i} - \sum_{b \leq a} \ddot{x}_{br}^i \frac{\partial}{\partial \ddot{x}_{b \leq a}^i}, \\ \mathcal{K}_{tt}^r = \frac{\partial}{\partial x^r} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_t^i} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_t^i} - 2 \ddot{x}_{tr}^i \frac{\partial}{\partial \ddot{x}_{tt}^i} - \ddot{x}_{ar}^i \frac{\partial}{\partial \ddot{x}_{ta}^i} - \ddot{x}_{ra}^i \frac{\partial}{\partial \ddot{x}_{ta}^i},$$

$$\begin{aligned}\mathcal{K}_{ta}^r = & \dot{x}^a \left(2 \frac{\partial}{\partial \ddot{x}^r} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_t^i} \right) + \ddot{x}^a \frac{\partial}{\partial \ddot{x}_t^r} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_a^i} + 2 \frac{\partial}{\partial \ddot{x}_a^r} + 2 \ddot{x}_t^a \frac{\partial}{\partial \ddot{x}_t^r} - 2 \dot{x}^a \ddot{x}_{tr}^i \frac{\partial}{\partial \ddot{x}_{tt}^i} \\ & - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_{ta}^i} + \ddot{x}_b^a \frac{\partial}{\partial \ddot{x}_{tb}^r} - \ddot{x}_{tr}^i \frac{\partial}{\partial \ddot{x}_{ta}^i} - \dot{x}^a \ddot{x}_{br}^i \frac{\partial}{\partial \ddot{x}_{tb}^i} + \ddot{x}_b^a \frac{\partial}{\partial \ddot{x}_{tb}^r} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_{ta}^i} \\ & - \dot{x}^a \ddot{x}_{rb}^i \frac{\partial}{\partial \ddot{x}_{tb}^i} - \ddot{x}_{rb}^i \frac{\partial}{\partial \ddot{x}_{a \leq b}^i} - \ddot{x}_{br}^i \frac{\partial}{\partial \ddot{x}_{b \leq a}^i} - \ddot{x}_{rb}^i \frac{\partial}{\partial \ddot{x}_{ab}^i},\end{aligned}$$

$$\begin{aligned}\mathcal{K}_{a \leq b}^r = & \frac{1}{1+\delta_{ab}} \left\{ 2 \dot{x}^a \dot{x}^b \frac{\partial}{\partial \ddot{x}^r} + \ddot{x}^b \frac{\partial}{\partial \ddot{x}_a^r} + \ddot{x}^a \frac{\partial}{\partial \ddot{x}_b^r} - \dot{x}^b \ddot{x}_r^k \frac{\partial}{\partial \ddot{x}_a^k} - \dot{x}^a \ddot{x}_r^k \frac{\partial}{\partial \ddot{x}_b^k} + 2 \dot{x}^b \frac{\partial}{\partial \ddot{x}_a^r} \right. \\ & + 2 \dot{x}^a \frac{\partial}{\partial \ddot{x}_b^r} + \ddot{x}_t^b \frac{\partial}{\partial \ddot{x}_t^r} + \ddot{x}_t^a \frac{\partial}{\partial \ddot{x}_t^r} - \dot{x}^b \ddot{x}_{tr}^i \frac{\partial}{\partial \ddot{x}_{ta}^i} - \dot{x}^a \ddot{x}_{tr}^i \frac{\partial}{\partial \ddot{x}_{tb}^i} + \ddot{x}_c^a \frac{\partial}{\partial \ddot{x}_{c \leq b}^r} \\ & + \ddot{x}_c^b \frac{\partial}{\partial \ddot{x}_{c \leq a}^r} + \ddot{x}_c^b \frac{\partial}{\partial \ddot{x}_{a \leq c}^r} + \ddot{x}_c^a \frac{\partial}{\partial \ddot{x}_{b \leq c}^r} - \dot{x}^b \ddot{x}_{rc}^i \frac{\partial}{\partial \ddot{x}_{a \leq c}^i} - \dot{x}^a \ddot{x}_{rc}^i \frac{\partial}{\partial \ddot{x}_{b \leq c}^i} \\ & - \dot{x}^a \ddot{x}_{cr}^i \frac{\partial}{\partial \ddot{x}_{c \leq b}^i} - \dot{x}^b \ddot{x}_{cr}^i \frac{\partial}{\partial \ddot{x}_{c \leq a}^i} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_{ab}^i} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_{ba}^i} + \ddot{x}_c^b \frac{\partial}{\partial \ddot{x}_{ac}^r} \\ & \left. + \ddot{x}_c^a \frac{\partial}{\partial \ddot{x}_{bc}^r} - \dot{x}^b \ddot{x}_{rc}^i \frac{\partial}{\partial \ddot{x}_{ac}^i} - \dot{x}^a \ddot{x}_{rc}^i \frac{\partial}{\partial \ddot{x}_{bc}^i} \right\} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_{a \leq b}^i} + 2 \frac{\partial}{\partial \ddot{x}_{a \leq b}^r},\end{aligned}$$

$$\begin{aligned}\mathcal{K}_{ttt}^r &= \frac{\partial}{\partial \ddot{x}_t^r} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_{tt}^i}, \\ \mathcal{K}_{tta}^r &= 2 \dot{x}^a \frac{\partial}{\partial \ddot{x}_t^r} + \frac{\partial}{\partial \ddot{x}_a^r} + \ddot{x}^a \frac{\partial}{\partial \ddot{x}_{tt}^r} - \dot{x}^a \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_{tt}^i} - \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_{ta}^i} + 2 \frac{\partial}{\partial \ddot{x}_{ta}^r},\end{aligned}$$

$$\begin{aligned}\mathcal{K}_{t,a \leq b}^r = & \frac{1}{1+\delta_{ab}} \left\{ 2 \dot{x}^a \dot{x}^b \frac{\partial}{\partial \ddot{x}_t^r} + 2 \dot{x}^b \frac{\partial}{\partial \ddot{x}_a^r} + 2 \dot{x}^a \frac{\partial}{\partial \ddot{x}_b^r} + \ddot{x}^b \frac{\partial}{\partial \ddot{x}_{ta}^r} + \ddot{x}^a \frac{\partial}{\partial \ddot{x}_{tb}^r} - \dot{x}^b \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_{ta}^i} \right. \\ & \left. - \dot{x}^a \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_{tb}^i} + 2 \dot{x}^b \frac{\partial}{\partial \ddot{x}_{ta}^r} + 2 \dot{x}^a \frac{\partial}{\partial \ddot{x}_{tb}^r} + 2 \frac{\partial}{\partial \ddot{x}_{ab}^r} + 2 \frac{\partial}{\partial \ddot{x}_{ba}^r} \right\} - \sum_{a \leq b} \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_{a \leq b}^i},\end{aligned}$$

$$\begin{aligned}\mathcal{K}_{tttt}^r &= \frac{\partial}{\partial \ddot{x}_{tt}^r}, \\ \mathcal{K}_{ttta}^r &= 2 \dot{x}^a \frac{\partial}{\partial \ddot{x}_{tt}^r} + \frac{\partial}{\partial \ddot{x}_{ta}^r}, \\ \mathcal{K}_{tt,a \leq b}^r &= \frac{1}{1+\delta_{ab}} \left(2 \dot{x}^a \dot{x}^b \frac{\partial}{\partial \ddot{x}_{tt}^r} + 2 \dot{x}^b \frac{\partial}{\partial \ddot{x}_{ta}^r} + 2 \dot{x}^a \frac{\partial}{\partial \ddot{x}_{tb}^r} \right) + \frac{\partial}{\partial \ddot{x}_{a \leq b}^r},\end{aligned}$$

$$\begin{aligned}\sum_{a \leq b \leq c} \frac{\partial^3 u^r}{\partial x^a \partial x^b \partial x^c} \mathcal{K}_{a \leq b \leq c}^r &= \sum_{a \leq b} \frac{\partial^3 u^r}{\partial x^a \partial x^b \partial x^c} \left(\dot{x}^h \frac{\partial}{\partial \ddot{x}_{a \leq b}^r} - \dot{x}^h \ddot{x}_r^i \frac{\partial}{\partial \ddot{x}_{a \leq b}^i} \right) \\ &+ \frac{\partial^3 u^r}{\partial x^a \partial x^b \partial x^c} \left(\dot{x}^b \ddot{x}_c^i \frac{\partial}{\partial \ddot{x}_a^i} + 2 \dot{x}^c \frac{\partial}{\partial \ddot{x}_{ab}^r} \right),\end{aligned}$$

$$\begin{aligned}\sum_{a \leq b \leq c} \frac{\partial^4 u^r}{\partial t \partial x^a \partial x^b \partial x^c} \mathcal{K}_{t,a \leq b \leq c}^r &= \frac{\partial^4 u^r}{\partial t \partial x^a \partial x^b \partial x^c} \dot{x}^c \dot{x}^b \frac{\partial}{\partial \ddot{x}_{ta}^r} + 2 \sum_{a \leq b} \frac{\partial^4 u^r}{\partial t \partial x^a \partial x^b \partial x^c} \dot{x}^c \frac{\partial}{\partial \ddot{x}_{a \leq b}^r}, \\ \sum_{a \leq b \leq c \leq d} \frac{\partial^4 u^r}{\partial x^a \partial x^b \partial x^c \partial x^d} \mathcal{K}_{a \leq b \leq c \leq d}^r &= \sum_{a \leq b} \frac{\partial^4 u^r}{\partial x^a \partial x^b \partial x^c \partial x^d} \dot{x}^c \dot{x}^d \frac{\partial}{\partial \ddot{x}_{a \leq b}^r}.\end{aligned}$$

From the expressions for \mathcal{K}_{tttt}^r , \mathcal{K}_{ttta}^r , $\mathcal{K}_{tt,a \leq b}^r$ above we deduce

$$\begin{aligned}\frac{\partial}{\partial \ddot{x}_{ta}^r} &= \mathcal{K}_{ttta}^r - 2 \dot{x}^a \mathcal{K}_{tttt}^r, \\ \frac{\partial}{\partial \ddot{x}_{a \leq b}^r} &= \mathcal{K}_{tt,a \leq b}^r - \frac{2}{1+\delta_{ab}} \left(\dot{x}^b \mathcal{K}_{ttta}^r + \dot{x}^a \mathcal{K}_{tttb}^r - 3 \dot{x}^a \dot{x}^b \mathcal{K}_{tttt}^r \right).\end{aligned}$$

Hence the vector fields $\mathcal{K}_{t,a \leq b \leq c}^r$ and $\mathcal{K}_{a \leq b \leq c \leq d}^r$ can be written as linear combinations of \mathcal{K}_{tttt}^r , \mathcal{K}_{ttta}^r , and $\mathcal{K}_{tt,a \leq b}^r$; namely,

$$\begin{aligned} \mathcal{K}_{t,a \leq b \leq c}^r &\equiv 0 \mod \left\langle \mathcal{K}_{tttt}^r, \mathcal{K}_{ttta}^r, \mathcal{K}_{tttb}^r, \mathcal{K}_{tttc}^r, \mathcal{K}_{tt,a \leq b}^r, \mathcal{K}_{tt,a \leq c}^r, \mathcal{K}_{tt,b \leq c}^r \right\rangle, \\ \mathcal{K}_{a \leq b \leq c \leq d}^r &\equiv 0 \mod \left\langle \mathcal{K}_{tttt}^r, \mathcal{K}_{ttta}^r, \mathcal{K}_{tttb}^r, \mathcal{K}_{tttc}^r, \mathcal{K}_{tttd}^r, \mathcal{K}_{tt,a \leq b}^r, \mathcal{K}_{tt,a \leq c}^r, \mathcal{K}_{tt,a \leq d}^r, \right. \\ &\quad \left. \mathcal{K}_{tt,b \leq c}^r, \mathcal{K}_{tt,b \leq d}^r, \mathcal{K}_{tt,c \leq d}^r \right\rangle, \end{aligned}$$

and, as a computation shows, the vector fields $\mathcal{K}_{a \leq b \leq c}^r$ can be written as linear combinations of \mathcal{K}_{tta}^r , $\mathcal{K}_{t,a \leq b}^r$, \mathcal{K}_{tttt}^r , \mathcal{K}_{ttta}^r , and $\mathcal{K}_{tt,a \leq b}^r$; namely,

$$\begin{aligned} \mathcal{K}_{a \leq b \leq c}^r &\equiv 0 \mod \left\langle \mathcal{K}_{ttt}^r, \mathcal{K}_{tta}^r, \mathcal{K}_{ttb}^r, \mathcal{K}_{ttc}^r, \mathcal{K}_{t,a \leq b}^r, \mathcal{K}_{t,a \leq c}^r, \mathcal{K}_{t,b \leq c}^r, \mathcal{K}_{tttt}^i, \mathcal{K}_{ttta}^i, \mathcal{K}_{tttb}^i, \right. \\ &\quad \left. \mathcal{K}_{tttc}^i, \mathcal{K}_{tt,a \leq b}^i, \mathcal{K}_{tt,a \leq c}^i, \mathcal{K}_{tt,b \leq c}^i \right\rangle. \end{aligned}$$

Moreover, from the previous formulas, we obtain

$$\begin{aligned} \mathcal{K}_t^r &= \frac{\partial}{\partial \bar{x}^r} - \ddot{x}_{r\dot{a}}^i \frac{\partial}{\partial \bar{x}_{t\dot{a}}^i} - \ddot{x}_r^i (\mathcal{K}_{ttt}^i + \ddot{x}_i^k \mathcal{K}_{tttt}^k) - 2\ddot{x}_{tr}^i \mathcal{K}_{tttt}^i - \ddot{x}_{ar}^i (\mathcal{K}_{ttta}^i - 2\dot{x}^a \mathcal{K}_{tttt}^i), \\ \mathcal{K}_{tt}^r &= \frac{\partial}{\partial \bar{x}^r} - \ddot{x}_{rb}^i \frac{\partial}{\partial \bar{x}_{tb}^i} - \ddot{x}_r^i (\mathcal{K}_{ttt}^i + \ddot{x}_i^k \mathcal{K}_{tttt}^k) - (\ddot{x}_r^i + 2\ddot{x}_{tr}^i) \mathcal{K}_{tttt}^i - \ddot{x}_{br}^i (\mathcal{K}_{tttb}^i - 2\dot{x}^b \mathcal{K}_{tttt}^i), \end{aligned}$$

$$\begin{aligned} \mathcal{K}_a^r &= \dot{x}^a \frac{\partial}{\partial \bar{x}^r} + \ddot{x}^a \frac{\partial}{\partial \bar{x}^r} + \ddot{x}_b^a \frac{\partial}{\partial \bar{x}_b^r} + \ddot{x}_b^a \frac{\partial}{\partial \bar{x}_b^r} + \ddot{x}_{tb}^a \frac{\partial}{\partial \bar{x}_{tb}^r} + \ddot{x}_{cb}^a \frac{\partial}{\partial \bar{x}_{cb}^r} + \sum_{c \leq b} \ddot{x}_{c \leq b}^a \frac{\partial}{\partial \bar{x}_{c \leq b}^r} \\ &\quad - \ddot{x}_r^i \frac{\partial}{\partial \bar{x}_a^i} - \ddot{x}_r^i \frac{\partial}{\partial \bar{x}_a^i} - \ddot{x}_{tr}^i \frac{\partial}{\partial \bar{x}_{ta}^i} - \ddot{x}_{rb}^i \frac{\partial}{\partial \bar{x}_{ab}^i} - \ddot{x}_{cr}^i \frac{\partial}{\partial \bar{x}_{ca}^i} - \sum_{a \leq b} \ddot{x}_{br}^i \frac{\partial}{\partial \bar{x}_{a \leq b}^i} \\ &\quad - \sum_{b \leq a} \ddot{x}_{br}^i \frac{\partial}{\partial \bar{x}_{b \leq a}^i} + \sum_{c \leq b} \ddot{x}_{cb}^a \left(\mathcal{K}_{tt,c \leq b}^r - \frac{2}{1+\delta_{cb}} (\dot{x}^b \mathcal{K}_{tttc}^r + \dot{x}^c \mathcal{K}_{tttb}^r - 3\dot{x}^c \dot{x}^b \mathcal{K}_{tttt}^r) \right) \\ &\quad - \sum_{c \leq a} \ddot{x}_{cr}^i \left(\mathcal{K}_{tt,c \leq a}^i - \frac{2}{1+\delta_{ca}} (\dot{x}^a \mathcal{K}_{tttc}^i + \dot{x}^c \mathcal{K}_{ttta}^i - 3\dot{x}^c \dot{x}^a \mathcal{K}_{tttt}^i) \right) \\ &\quad - \sum_{a \leq b} \ddot{x}_{rb}^i \left(\mathcal{K}_{tt,a \leq b}^i - \frac{2}{1+\delta_{ab}} (\dot{x}^a \mathcal{K}_{tttb}^i + \dot{x}^b \mathcal{K}_{ttta}^i - 3\dot{x}^b \dot{x}^a \mathcal{K}_{tttt}^i) \right) \\ &\quad + \ddot{x}_t^a (\mathcal{K}_{ttt}^r + \ddot{x}_r^k \mathcal{K}_{tttt}^k) + \ddot{x}_{tt}^a \mathcal{K}_{tttt}^r - \ddot{x}_{tr}^i (\mathcal{K}_{ttta}^i - 2\dot{x}^a \mathcal{K}_{tttt}^i) \\ &\quad + \ddot{x}_{tb}^a (\mathcal{K}_{tttb}^r - 2\dot{x}^b \mathcal{K}_{tttt}^r), \end{aligned}$$

$$\begin{aligned} \mathcal{K}_{ta}^r &= \dot{x}^a (\mathcal{K}_{tt}^r + \ddot{x}_r^i \mathcal{K}_{tttt}^i) + \ddot{x}^a (\mathcal{K}_{ttt}^r + \ddot{x}_r^i \mathcal{K}_{tttt}^i) + 2\ddot{x}_t^a \mathcal{K}_{tttt}^r - \ddot{x}_r^i (\mathcal{K}_{ttta}^i - 2\dot{x}^a \mathcal{K}_{tttt}^i) \\ &\quad + \ddot{x}_b^a (\mathcal{K}_{tttb}^r - 2\dot{x}^b \mathcal{K}_{tttt}^r) + \dot{x}^a \frac{\partial}{\partial \bar{x}^r} - \ddot{x}_r^i \frac{\partial}{\partial \bar{x}_a^i} + 2\frac{\partial}{\partial \bar{x}_a^r} - \ddot{x}_{rc}^i \frac{\partial}{\partial \bar{x}_{ac}^i} - \ddot{x}_r^i \frac{\partial}{\partial \bar{x}_{ta}^i} + \ddot{x}_b^a \frac{\partial}{\partial \bar{x}_{tb}^i} \\ &\quad - \ddot{x}_{tr}^i (\mathcal{K}_{ttta}^i - 2\dot{x}^a \mathcal{K}_{tttt}^i) - \ddot{x}_{rc}^i \left(\mathcal{K}_{tt,a \leq c}^i - \frac{2}{1+\delta_{ac}} (\dot{x}^c \mathcal{K}_{ttta}^i + \dot{x}^a \mathcal{K}_{tttc}^i - 3\dot{x}^a \dot{x}^c \mathcal{K}_{tttt}^i) \right) \\ &\quad - \ddot{x}_{cr}^i \left(\mathcal{K}_{tt,c \leq a}^i - \frac{2}{1+\delta_{ac}} (\dot{x}^c \mathcal{K}_{ttta}^i + \dot{x}^a \mathcal{K}_{tttc}^i - 3\dot{x}^a \dot{x}^c \mathcal{K}_{tttt}^i) \right), \end{aligned}$$

$$\begin{aligned}
\kappa_{a \leq b}^r = & \frac{1}{1+\delta_{ab}} \left[\ddot{x}^b \frac{\partial}{\partial \ddot{x}_a^r} + \ddot{x}^a \frac{\partial}{\partial \ddot{x}_b^r} + \ddot{x}_r^i \left(\dot{x}^b \frac{\partial}{\partial \ddot{x}_{ta}^i} + \dot{x}^a \frac{\partial}{\partial \ddot{x}_{tb}^i} - \frac{\partial}{\partial \ddot{x}_{ab}^i} - \frac{\partial}{\partial \ddot{x}_{ba}^i} \right) \right. \\
& + \ddot{x}_c^a \left(\frac{\partial}{\partial \ddot{x}_{bc}^r} - \dot{x}^b \frac{\partial}{\partial \ddot{x}_{tc}^r} \right) + \ddot{x}_c^b \left(\frac{\partial}{\partial \ddot{x}_{ac}^r} - \dot{x}^a \frac{\partial}{\partial \ddot{x}_{tc}^r} \right) - \dot{x}^b \left\{ \dot{x}^a (\kappa_{tt}^r + \ddot{x}_r^i \kappa_{tttt}^i) \right. \\
& + \ddot{x}^a (\kappa_{ttt}^r + \ddot{x}_r^i \kappa_{tttt}^i) + 2\ddot{x}_t^a \kappa_{tttt}^r - \ddot{x}_r^i (\kappa_{ttta}^i - 2\dot{x}^a \kappa_{tttt}^i) \left. \right\} \\
& + \ddot{x}_c^a (\kappa_{tttc}^r - 2\dot{x}^c \kappa_{tttt}^r) + \dot{x}^b \kappa_{ta}^r + \dot{x}^a \kappa_{tb}^r \\
& - \dot{x}^a \left\{ \dot{x}^b (\kappa_{tt}^r + \ddot{x}_r^i \kappa_{tttt}^i) + \dot{x}^b (\kappa_{ttt}^r + \ddot{x}_r^i \kappa_{tttt}^i) + 2\dot{x}_t^b \kappa_{tttt}^r \right. \\
& - \ddot{x}_r^i (\kappa_{tttb}^i - 2\dot{x}^b \kappa_{tttt}^i) + \ddot{x}_c^b (\kappa_{tttc}^r - 2\dot{x}^c \kappa_{tttt}^r) \left. \right\} \\
& + \ddot{x}_t^b (\kappa_{ttta}^r - 2\dot{x}^a \kappa_{tttt}^r) + \ddot{x}_t^a (\kappa_{tttb}^r - 2\dot{x}^b \kappa_{tttt}^r) \\
& + \ddot{x}_c^a \left(\kappa_{tt,c \leq b}^r - \frac{2}{1+\delta_{cb}} (\dot{x}^b \kappa_{tttc}^r + \dot{x}^c \kappa_{tttb}^r - 3\dot{x}^c \dot{x}^b \kappa_{tttt}^r) \right) \\
& + \ddot{x}_c^b \left(\kappa_{tt,c \leq a}^r - \frac{2}{1+\delta_{ca}} (\dot{x}^a \kappa_{tttc}^r + \dot{x}^c \kappa_{ttta}^r - 3\dot{x}^c \dot{x}^a \kappa_{tttt}^r) \right) \\
& + \ddot{x}_c^a \left(\kappa_{tt,a \leq c}^r - \frac{2}{1+\delta_{ca}} (\dot{x}^a \kappa_{tttc}^r + \dot{x}^c \kappa_{ttta}^r - 3\dot{x}^c \dot{x}^a \kappa_{tttt}^r) \right) \\
& + \ddot{x}_c^b \left(\kappa_{tt,b \leq c}^r - \frac{2}{1+\delta_{cb}} (\dot{x}^b \kappa_{tttc}^r + \dot{x}^c \kappa_{tttb}^r - 3\dot{x}^c \dot{x}^b \kappa_{tttt}^r) \right) \left. \right] \\
& - \ddot{x}_r^i \left(\kappa_{tt,a \leq b}^i - \frac{1}{1+\delta_{ab}} (2\dot{x}^b \kappa_{ttta}^i + 2\dot{x}^a \kappa_{tttb}^i - 6\dot{x}^a \dot{x}^b \kappa_{tttt}^i) \right) + 2 \frac{\partial}{\partial \ddot{x}_{a \leq b}^r},
\end{aligned}$$

$$\kappa_{ttt}^r = \frac{\partial}{\partial \ddot{x}_t^r} - \ddot{x}_r^i \kappa_{tttt}^i,$$

$$\kappa_{tta}^r = 2\dot{x}^a \kappa_{ttt}^r + \ddot{x}^a \kappa_{tttt}^r + 3\dot{x}^a \ddot{x}_r^i \kappa_{tttt}^i - \ddot{x}_r^i \kappa_{ttta}^i + \frac{\partial}{\partial \ddot{x}_a^r} + 2 \frac{\partial}{\partial \ddot{x}_{ta}^r},$$

$$\begin{aligned}
\kappa_{t,a \leq b}^r = & \frac{1}{1+\delta_{ab}} \left\{ \dot{x}^b (\kappa_{tta}^r - \dot{x}^a \kappa_{ttt}^r - \ddot{x}^a \kappa_{tttt}^r) + \dot{x}^a (\kappa_{ttb}^r - \dot{x}^b \kappa_{ttt}^r - \ddot{x}^b \kappa_{tttt}^r) \right. \\
& + \ddot{x}^b (\kappa_{ttta}^r - 2\dot{x}^a \kappa_{tttt}^r) + \ddot{x}^a (\kappa_{tttb}^r - 2\dot{x}^b \kappa_{tttt}^r) + \dot{x}^b \frac{\partial}{\partial \ddot{x}_a^r} + \dot{x}^a \frac{\partial}{\partial \ddot{x}_b^r} \\
& + 2 \frac{\partial}{\partial \ddot{x}_{ab}^r} + 2 \frac{\partial}{\partial \ddot{x}_{ba}^r} \left. \right\} \\
& - \sum_{a \leq b} \ddot{x}_r^i \left(\kappa_{tt,a \leq b}^i - \frac{2}{1+\delta_{ab}} (\dot{x}^b \kappa_{ttta}^i + \dot{x}^a \kappa_{tttb}^i - 3\dot{x}^a \dot{x}^b \kappa_{tttt}^i) \right),
\end{aligned}$$

$$\kappa_{tttt}^r = \frac{\partial}{\partial \ddot{x}_{tt}^r},$$

$$\kappa_{ttta}^r = 2\dot{x}^a \kappa_{tttt}^r + \frac{\partial}{\partial \ddot{x}_{ta}^r},$$

$$\kappa_{tt,a \leq b}^r = \frac{2}{1+\delta_{ab}} (\dot{x}^b \kappa_{ttta}^r + \dot{x}^a \kappa_{tttb}^r - 3\dot{x}^a \dot{x}^b \kappa_{tttt}^r) + \frac{\partial}{\partial \ddot{x}_{a \leq b}^r}.$$

Hence for $n \geq 2$,

$$\mathcal{D}^{(2)} = \langle \kappa_{00}^r, \kappa_t^r, \kappa_a^r, \kappa_{tt}^r, \kappa_{ta}^r, \kappa_{a \leq b}^r, \kappa_{ttt}^r, \kappa_{tta}^r, \kappa_{t,a \leq b}^r, \kappa_{tttt}^r, \kappa_{ttta}^r, \kappa_{tt,a \leq b}^r \rangle$$

$$\text{and for } n = 1, \mathcal{D}^{(2)} = \langle \kappa_{00}, \kappa_t, \kappa_{tt}, \kappa_{t1}, \kappa_{11}, \kappa_{ttt}, \kappa_{tt1}, \kappa_{t,11}, \kappa_{tttt}, \kappa_{ttt1}, \kappa_{tt,11} \rangle. \quad \square$$

According to the formulas (14), (15), (16) the tensor field K^σ takes values in the vector bundle $(p^{10})^* \wedge^2 T^* M^0 \otimes V(p^{10})$ and we can define the curvature mapping $\mathcal{K}: J^2(p^{21}) \rightarrow (p^{10})^* \wedge^2 T^* M^0 \otimes V(p^{10})$ by setting $\mathcal{K}(j_\xi^2 \sigma) = (K^\sigma)_\xi$.

Theorem 8.2. *Every second-order differential invariant $\mathcal{I}: J^2(p^{21}) \rightarrow \mathbb{R}$ with respect to $\text{Aut}^v(p)$ factors uniquely through the curvature mapping as follows: $\mathcal{I} = \tilde{\mathcal{I}} \circ \mathcal{K}$, where $\tilde{\mathcal{I}}: (p^{10})^* \bigwedge^2 T^*M^0 \otimes V(p^{10}) \rightarrow \mathbb{R}$ is an invariant smooth function under the natural action of $\text{Aut}^v(p)$, namely,*

$$(74) \quad \Phi \cdot \eta = \left(\bigwedge^2 ((\Phi^{(1)})^{-1})^* \otimes (\Phi^{(1)})_* \right) (\eta), \quad \forall \eta \in (p^{10})^* \bigwedge^2 T^*M^0 \otimes V(p^{10}).$$

Proof. The statement is an immediate consequence of the following properties:

1. The curvature mapping is a surjective submersion.
2. The fibre $\mathcal{K}^{-1}(\eta)$ for every $\eta \in \bigwedge^2 T_{(t_0, x_0)}^* M^0 \otimes V_\xi(p^{10})$ is an affine sub-bundle over $J^1(p^{21})$; in particular, the fibres of $J^2(p^{21})$ are connected.
3. If we define

$$\tilde{\mathcal{D}}_{j_\xi^2 \sigma}^{(2)} = \begin{cases} \left\{ (X^{(2)})_{j_\xi^2 \sigma}^{(2)} \in \mathcal{D}_{j_\xi^2 \sigma}^{(2)} : j_{j_{t_0}^1 \gamma}^1 X = 0 \right\}, & \text{if } n = \dim M \geq 2 \\ \left\{ (X^{(2)})_{j_\xi^2 \sigma}^{(2)} \in \mathcal{D}_{j_\xi^2 \sigma}^{(2)} : X_\xi^{(1)} = 0 \right\}, & \text{if } n = \dim M = 1 \end{cases}$$

$$\text{then } \ker(\mathcal{K}_*)_{j_\xi^2 \sigma} = \tilde{\mathcal{D}}_{j_\xi^2 \sigma}^{(2)}, \quad \xi = j_{t_0}^1 \gamma.$$

4. The curvature mapping is $\text{Aut}^v(p)$ -equivariant with respect to the natural actions, i.e., $\Phi \cdot \mathcal{K}(j_\xi^2 \sigma) = \mathcal{K}(\Phi \cdot j_\xi^2 \sigma)$, where the action on the left-hand side is defined in (74) and that on the right-hand side is given as in the beginning of this section, i.e., $\Phi \cdot j_\xi^2 \sigma = j_{\Phi^{(1)}(\xi)}^2 (\Phi^{(2)} \circ \sigma \circ (\Phi^{(1)})^{-1})$.

Coordinates are introduced in $(p^{10})^* \bigwedge^2 T^*M^0 \otimes V(p^{10})$ by setting

$$\eta = \left\{ y_i^j(\eta) (dt \wedge \omega^i)_{(t_0, x_0)} + \sum_{h < i} y_{hi}^j(\eta) (\omega^h \wedge \omega^i)_{(t_0, x_0)} \right\} \otimes \left(\frac{\partial}{\partial \dot{x}^j} \right)_\xi,$$

for every $\eta \in \bigwedge^2 T_{(t_0, x_0)}^* M^0 \otimes V_\xi(p^{10})$. The first and second properties directly follow from the equations of the curvature mapping, i.e.,

$$\begin{aligned} t \circ \mathcal{K} &= t, \quad x^i \circ \mathcal{K} = x^i, \quad \dot{x}^i \circ \mathcal{K} = \dot{x}^i, \\ y_a^i \circ \mathcal{K} &= -\frac{1}{2} \ddot{x}_{t\dot{a}}^i - \frac{1}{2} \dot{x}^h \ddot{x}_{h\dot{a}}^i - \frac{1}{2} \ddot{x}^h \ddot{x}_{h\dot{a}}^i + \ddot{x}_a^i + \frac{1}{4} \ddot{x}_a^k \ddot{x}_{\dot{k}}^i, \\ y_{ab}^k \circ \mathcal{K} &= -\frac{1}{2} \ddot{x}_{a\dot{b}}^k + \frac{1}{2} \ddot{x}_{b\dot{a}}^k - \frac{1}{4} \ddot{x}_a^h \ddot{x}_{h\dot{b}}^k + \frac{1}{4} \ddot{x}_b^h \ddot{x}_{h\dot{a}}^k, \quad a < b. \end{aligned}$$

Moreover, from the formulas (67)–(73) we deduce

$$\begin{aligned} (X^{(2)})^{(2)} (y_j^i \circ \mathcal{K}) &= \frac{\partial u^i}{\partial x^r} (y_j^r \circ \mathcal{K}) - \frac{\partial u^r}{\partial x^j} (y_r^i \circ \mathcal{K}), \\ (X^{(2)})^{(2)} (y_{ij}^k \circ \mathcal{K}) &= \frac{\partial u^r}{\partial x^i} (y_{rj}^k \circ \mathcal{K}) + \frac{\partial u^k}{\partial x^r} (y_{ji}^r \circ \mathcal{K}) + \frac{\partial u^r}{\partial x^j} (y_{ir}^k \circ \mathcal{K}). \end{aligned}$$

By evaluating these two formulas at $j_\xi^2 \sigma$, we conclude $\tilde{\mathcal{D}}_{j_\xi^2 \sigma}^{(2)} \subseteq \ker(\mathcal{K}_*)_{j_\xi^2 \sigma}$ and from the Lemma 8.1 and the first item above we have

$$\begin{aligned} \dim \ker(\mathcal{K}_*)_{j_\xi^2 \sigma} &= \dim J^2(p^{21}) - \dim \left((p^{10})^* \bigwedge^2 T^* M^0 \otimes V(p^{10}) \right) \\ &= \frac{3}{2}n(n+2)(n+1) \\ &= \dim \tilde{\mathcal{D}}_{j_\xi^2 \sigma}^{(2)}. \end{aligned}$$

Finally, the fourth item above follows by using the formulas (55)–(63) and the fact that $T^{\Phi \cdot \sigma}(\Phi^{(1)} \cdot X, \Phi^{(1)} \cdot Y) = \Phi^{(1)} \cdot T^\sigma(X, Y)$, $\forall X, Y \in \mathfrak{X}(M^1)$, as follows from Corollary 7.2. \square

Remark 8.3. As the distribution $\mathcal{D}^{(2)}$ is involutive, the number of functionally independent second-order differential invariants is $\frac{1}{2}n^2(n-1) + 1$ if $n \geq 2$, and 2 if $n = 1$. By passing to the quotient, the isomorphism (10) induces another isomorphism

$$\begin{aligned} \iota_1: T^-(M^1) &\xrightarrow{\cong} (p^{10})^* TM^0 / (p^{10})_* T^0(M^1), \\ \iota_1(X_i^\sigma) &= \frac{\partial}{\partial x^i} \bmod (p^{10})_* T^0(M^1). \end{aligned}$$

Moreover, as $M^0 = \mathbb{R} \times M$, there is a natural embedding $(p')^* TM \hookrightarrow TM^0$ and pulling it back via p^{10} we obtain another embedding $(p' \circ p^{10})^* TM \hookrightarrow (p^{10})^* TM^0$. By composing this latter embedding and the quotient map

$$(p^{10})^* TM^0 \rightarrow (p^{10})^* TM^0 / (p^{10})_* T^0(M^1),$$

an isomorphism $\iota_2: (p' \circ p^{10})^* TM \xrightarrow{\cong} (p^{10})^* TM^0 / (p^{10})_* T^0(M^1)$ is deduced. From the formula (14) it follows $i_{X^\sigma} K^\sigma = -P_j^h \omega^j \otimes \partial / \partial \dot{x}^h$, and we define an endomorphism $\tilde{K}^\sigma: (p' \circ p^{10})^* TM \rightarrow (p' \circ p^{10})^* TM$ by setting

$$\begin{aligned} \tilde{K}^\sigma &= \varepsilon^{-1} \circ i_{X^\sigma} K^\sigma|_{T^-(M^1)} \circ (\iota_1)^{-1} \circ \iota_2, \\ \tilde{K}^\sigma \left(\frac{\partial}{\partial x^j} \right) &= -P_j^h \frac{\partial}{\partial \dot{x}^h}. \end{aligned}$$

The coefficients of the characteristic polynomial of \tilde{K}^σ determine n second-order invariants. This fact was remarked for the first time in [19]. If $n = 1$ or $n = 2$ then these invariants together with the function t exhaust a basis of second-order invariants, but this is no longer true for $n \geq 3$.

Finally, we should also like to remark that $\mathcal{K}(j_\xi^2 \sigma)$ depends only on $j_\xi^1(\tilde{K}^\sigma)$, as follows from the following identities among the components of the torsion tensor field of the Chern connection:

$$3T_{kj}^i = \frac{\partial P_j^i}{\partial \dot{x}^k} - \frac{\partial P_k^i}{\partial \dot{x}^j}.$$

Therefore, the Kosambi tensor field \tilde{K}^σ encodes all the relevant information for KCC theory.

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